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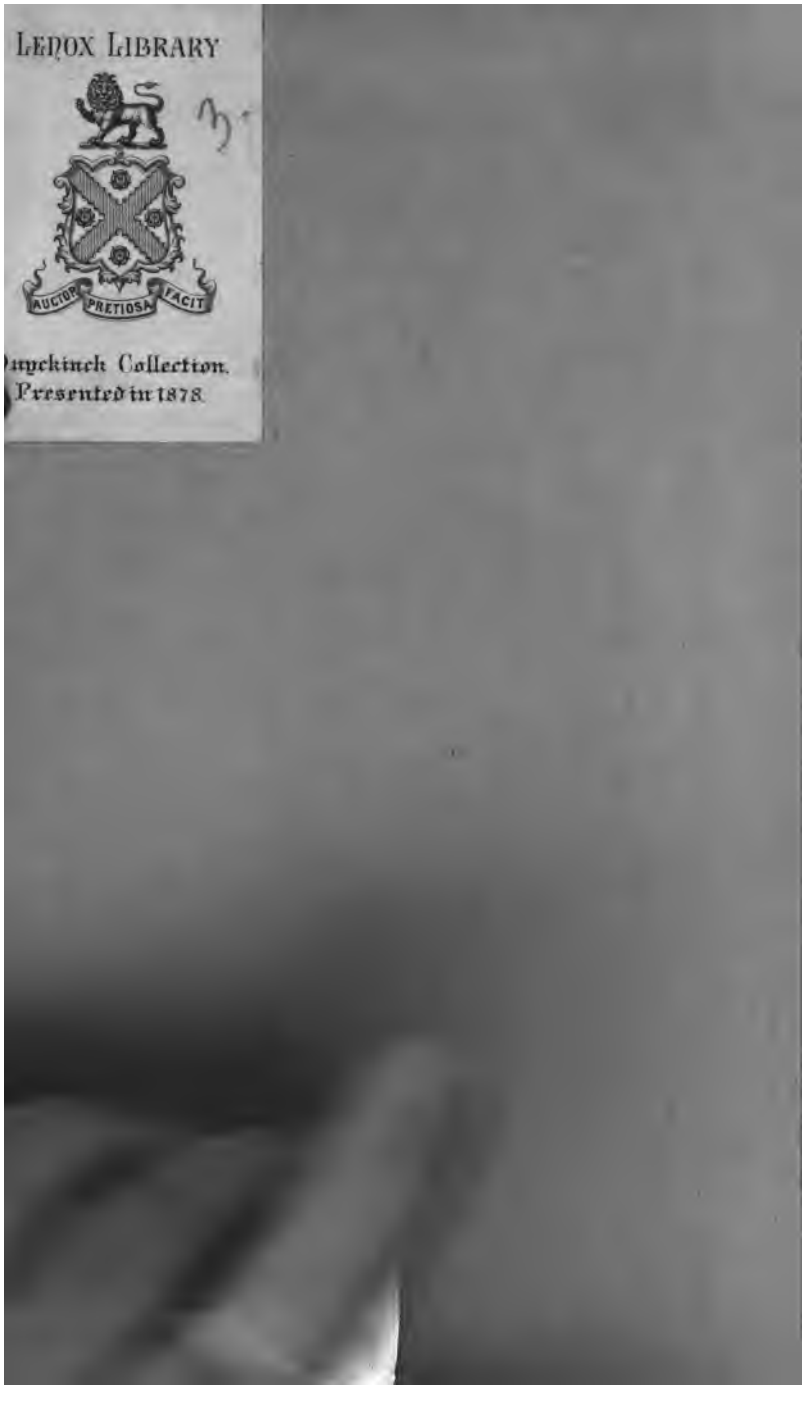

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UNIVERSITY ALGEBRA:

EMBRACING

A LOGICAL DEVELOPMENT OF THE SCIENCE,

WITH

NUMEROUS GRADED EXAMPLES.

BY 1798-1876

CHARLES DAVIES, LL.D.,

PROFESSOR OF MATHEMATICS IN COLUMBIA COLLEGE.

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PREFACE.

THE multiplication of Mathematical Text-Books, within the past few years, may be thought to have exceeded the demands of Science, and to have retarded, rather than advanced, the interests of Education. But this may not be so.

A text-book should be the living mind of its author, to be read and studied by those whom he would instruct. Every full and complete system of instruction is made up of many parts, wisely adjusted and carefully balanced. Hence, each part must bear a due proportion to all the others; and should, as far as possible, be developed after the same model.

The Mathematical Department of Columbia College has been confided to a Professor and an Adjunct Professor; and the duties of instruction have been divided between them. Under this arrangement, the College is entitled to their joint labors—not only in the lecture-room, but also, in the preparation of suitable text-books.

Mr. Peck, the Adjunct Professor of Mathematics, has been so kind as to place at my disposal all the methods of elementary instruction in Algebra, which he has wrought out and employed, with great success, for many years. Those used in the development of the Binomial Formula are deemed worthy of special notice.

It is the design of this work, so to treat the science of Algebra, that its peculiar logic may be strikingly exhibited in the harmonious combination of its principles, to the end that mental training may result from their contemplation; and to show, at the same time, the application of those principles in a carefully arranged series of graded Examples.

Columbia College,
New York, *June*, 1858.

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ELEMENTS OF ALGEBRA.

INTRODUCTION.

1. QUANTITY is anything that can be measured.

To measure a thing, is to find how many times it contains some other thing of the same kind taken as a standard. The assumed standard is called the *unit of measure.* A quantity may be measured by a direct application of the unit of measure, or the measurement may be made indirectly by comparing it with some other quantity whose measure is known.

2. The unit of measure being always of the same kind as the thing measured, there must be as many kinds of quantity as there are species of units.

In pure Mathematics there are but eight species of units, and consequently but eight kinds of quantity, viz.: Units of *Number*, Units of *Currency*, Units of *Weight*, Units of *Time*, Units of *Length*, Units of *Surface*, Units of *Volume*, and Units of *Angular Measure.* The first four kinds of quantity are *Arithmetical*, the last four are *Geometrical;* and these two classes form the entire subject of mathematical inquiry. By comparing quantities with each other, we arrive at a knowledge of their properties and relations.

3. MATHEMATICS is the science which treats of the properties and relations of quantities.

4. The conventional methods of representing quantities, and the laws for combining the symbols employed, constitute the LANGUAGE of Mathematics. Every branch of Mathematics has its own special language, which is but a dialect of what may be called the universal language of Mathematics.

5. The Science of Mathematics is divided into three principal branches, the basis of classification being the nature of the language employed. These branches are:

1*st.* ARITHMETIC, in which the quantities considered are expressed by figures and combinations of figures;

2*d.* ANALYSIS, in which the quantities considered are represented by letters, or by combinations of letters and figures, and in which the operations to be performed are indicated by signs;

3*d.* GEOMETRY, in which the quantities considered are represented by pictorial symbols.

The methods of reasoning are essentially the same in each branch, the language alone being different.

By combining the three branches, Arithmetic, Analysis, and Geometry, all the subordinate branches of mathematics are obtained.

6. ALGEBRA is a branch of Analysis; it is, indeed, the foundation of all the other branches of Analysis and its applications.

CHAPTER I.

DEFINITIONS AND EXPLANATION OF SIGNS.

1. ALGEBRA is a branch of Mathematics in which the quantities considered are represented by letters, and the operations to be performed are indicated by signs.

2. The quantities employed in Algebra are of two kinds:

1*st.* *Known Quantities*, those whose values are given; and

2*d.* *Unknown Quantities*, those whose values are required.

Known Quantities are generally represented by the leading letters of the alphabet; as, a, b, c, &c.

Unknown Quantities are generally represented by the final letters of the alphabet; as, x, y, z, w, &c.

When, in the course of an operation, an unknown quantity becomes known, it is often found convenient to represent it by one of the final letters, with one or more accents, as, x', y'', z''', &c. These symbols are read, x' *prime*, y'' *second*, z''' *third*, &c.

3. The signs employed in Algebra are of three kinds: signs of operation, signs of relation, and signs of abbreviation.

4. *The Signs of Operation are the following:*

1*st.* The *Sign of Addition*, $+$, called *plus*. When placed between two quantities, it indicates that the second is to be added to the first. Thus, the combination, $a + b$, read, *a plus b*, indicates that b is to be added to a.

2*d.* *Sign of Subtraction*, $-$, called *minus*. When placed between two quantities, it indicates that the second is to be subtracted from the first. Thus, the expression, $c - d$, read *c minus d*, indicates that d is to be subtracted from c.

3*d.* The *Sign of Multiplication*, $\times$. When placed between two quantities, it indicates that the first is to be multiplied by the second. Thus, the expression, $x \times y$, indicates that x is to be multiplied by y. Instead of the sign of multiplication, a simple point is sometimes used; or, when the factors are letters, they may be written one after the other, without any intervening sign. Thus, the expressions, $a^2 \times b \times c$, $a^2.b.c$, and a^2bc, have all the same meaning. A *factor*, is any one of the multipliers of a product.

4*th.* The *Sign of Division*, $\div$. When written between two quantities, it indicates that the first is to be divided by the second. Thus, the expression, $p \div q$, indicates that p is to be divided by q. The operation may also be expressed by writing one quantity over the other, in the form of a fraction; or the sign of division may be replaced by a straight or curved line. Thus, the expressions, $p \div q$, $\frac{p}{q}$, and $p \mid \underline{q}$, are all equivalent.

5*th.* The *Exponential Sign.* The exponential sign is a number written at the right and above a quantity, and indicates the number of times which the quantity is to be

taken as a factor. In the expressions, a^2, a^3, a^5, a^m, the numbers 2, 3, 5, m, are exponents, and indicate severally, that the quantity a is to be taken, 2, 3, 5, m times as a factor. The resulting products are called *powers.*

6*th.* The *Radical Sign*, $\sqrt{\ }$. When placed over a quantity, it indicates that a root of that quantity is to be extracted. The nature of the root is indicated by a number placed over the radical sign, called an *index.* Thus, the expressions, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[n]{a}$, indicate that the *square*, *cube*, and n^{th} *roots* of a, are to be extracted.

When no exponent is written over a quantity, the exponent 1 is always understood. When no index is written over the radical sign, the index 2 is understood.

7*th.* The sign $+$, is sometimes called the *positive* sign, and quantities before which it is written are called *positive quantities*, or *additive quantities.*

8*th.* The sign $-$, is sometimes called the *negative* sign, and quantities before which it is written are called *negative quantities*, or *subtractive quantities.*

9*th.* The *Double Sign* $\pm$, read *plus and minus*, is often used to indicate that the quantity before which it is placed, is first to be added to, and then to be subtracted from the preceding quantity. Thus, the expression, $p \pm \sqrt{q}$, is equivalent to the expressions, $p + \sqrt{q}$ and $p - \sqrt{q}$.

When no sign is written before a quantity, the sign $+$ is understood.

5. *The Signs of Relation are the following:*

1*st.* The *Sign of Equality*, $=$. When written between two quantities, it indicates that they are equal to

each other. Thus, the expression, $a = nb$, indicates that a is equal to the product of n and b.

2*d*. The *Sign of Inequality*, $<$, $>$. When written between two quantities, it indicates that they are unequal, the greater one being placed at the opening of the sign. Thus, the expressions, $a > b$, $c < d$, indicate that a *is greater than* b, and that c *is less than* d.

3*d*. The *Signs of Proportion*, : :: ∴ The single colon stands for, *is to*; the double colon for, *as*. Thus, the expression,

$$a : b :: c : d,$$

is read, a *is to* b, *as* c *is to* d.

4*th*. The *Sign of Variation*, $\propto$. When written between two quantities, it indicates that they increase or diminish together. Thus, the expression, $a \propto \frac{b}{c}$, indicates that a varies as $\frac{b}{c}$; that is, that a and $\frac{b}{c}$ increase or decrease together.

6. *The Signs of Abbreviation are the following:*

1*st*. The Sign ∴ stands for *hence*, or *consequently*.

2*d*. The Sign ∵ stands for *because*.

3*d*. The *Vinculum*, ———, the *Bar*, |, the *Parenthesis*, ( ), the *Brackets*, [ ], { }, are employed to connect several quantities, all of which are to be operated upon in the same manner. Thus, the expressions,

$$\overline{a+b+c} \times x, \qquad \left.\begin{matrix} a \\ +b \\ +c \end{matrix}\right| x, \qquad (a+b+c) \times x,$$

$$[a+b+c] \times x, \qquad \{a+b+c\} \times x,$$

all indicate that the sum of a, b, and c, is to be multiplied by x.

7. A *Coefficient* is a number written before a quantity, to show how many times it is to be taken additively. Thus, the expression, $3a^2$, is equivalent to $a^2 + a^2 + a^2$. The number 3 is the coefficient of a^2. The coefficient may be either *numerical* or *literal*. Thus, in the expressions, $3x^2$, $3ax^2$, $(a + b + c)x^2$, the quantities 3, $3a$, and $(a + b + c)$, are coefficients of x^2. When no coefficient is written, the coefficient 1 is understood.

8. A *Term* is an expression of a single quantity. Thus, a^2, $2a^2b$, $-\frac{7}{3}ax^3$, are terms.

9. A *Monomial* is a single term, unconnected with any other by the signs + or −. Thus, $3a^2$, $7a^2bc$, are monomials.

10. A *Polynomial* is a collection of terms connected by the signs + or −. Thus, $3a^2b + c - d$, $\frac{2ab - c}{3} + d$, are polynomials.

11. A *Binomial* is a polynomial of two terms; as,

$$c + d, \qquad (e + f)x, \qquad \frac{a}{b}x + y.$$

12. A *Trinomial* is a polynomial of three terms; as,

$$a + b + c, \quad x^2 - 2xy + y^2, \quad -a + 3b^3 + x.$$

The *degree* of a term is the number of its literal factors. Thus, the term $3a$ is of the first degree, because it contains one literal factor; the term $-7a^2$ is of the second degree, because it contains two literal factors; the term $-a^2bx^3$ is of the sixth degree, because it contains six literal factors. The degree of a term is determined by the sum of the exponents of all its letters.

Two terms are *homogeneous*, when they are of the

same degree. Thus, the terms $8a^2bx$ and y^3z are homogeneous, as are the terms a^3b^2c and $7x^3y^3$.

13. A *Polynomial is homogeneous* when all of its terms are homogeneous. Thus, the polynomial $a^3bc - 7ax^4 + 3b^3c^2$ is homogeneous, whilst the polynomial $8a^3b - 7a^2c + 8x^2$ is not homogeneous.

Two terms are *similar* or *like*, when the combination of literal factors is the same in both. Thus, the terms $8x^2yz$ and $-7x^2yz$, are similar, as are also the terms $25a^2bcd^3$ and $2a^2bcd^3$. In order that two terms may be similar, they must contain the same letters, and each letter must have the same exponent in both. The terms $8a^2b$ and $-7ab^2$ contain the same letters, but are not similar.

14. The *Reciprocal* of a quantity is 1 divided by that quantity: thus, $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{c}{d}$, x, are reciprocals of the quantities a, $a+b$, $\frac{d}{c}$ and $\frac{1}{x}$ respectively. The product of any quantity by its reciprocal, is equal to 1. Thus, $ab \times \frac{1}{ab}$ is equal to 1.

15. The *Numerical Value* of an expression, is the result obtained by assigning numerical values to each letter which enters it, and then performing all the operations indicated. Thus, the numerical value of the expression,

$$ab + ac + bc,$$

when $a = 2$, $b = 3$, and $c = 4$, is

$$2 \times 3 + 2 \times 4 + 3 \times 4 = 26;$$

by performing the operations indicated.

EXAMPLES.

Find the numerical values of the following expressions, when $a = 2$, $b = 3$, $c = 4$, and $d = 5$.

1. $ab + cb$. *Ans.* 18.
2. $ad - d + b$. *Ans.* 8.
3. $bc + ab - c$. *Ans.* 14.
4. $(bc + a)\ b$. *Ans.* 42.
5. $(bd - a)\ (ac - d)$. *Ans.* 39.
6. $(d + c)\ (d - c)$. *Ans.* 9.
7. $\frac{a + 2b}{c} + d$. *Ans.* 7.
8. $(a^2 - b)\ (c + d)$. *Ans.* 9.
9. $abc + cd + ad$. *Ans.* 54.
10. $\frac{a + bc}{7} \times (c + d)$. *Ans.* 18.

Find the numerical values of the following expressions, when $a = 5$, $b = 2$, $c = 4$, and $d = 3$.

11. $\frac{6}{a} - \frac{3}{b} + \frac{10}{c - d} - \frac{14}{c + d}$. *Ans.* $7\frac{7}{10}$.
12. $\left(\frac{a^2b}{c} \times d\right) \div \left(\frac{ab^2}{c} + d\right)$. *Ans.* 4.6875.
13. $\frac{a^2b^2c^2d^2 + 2abcd + 1}{abcd + 1}$. *Ans.* 121.
14. $\frac{a^2 + b^2 - d^2}{a + b + d} + \frac{abcd}{2b + c} - \frac{4a^2 - 10bc + 2}{2c + d}$. *Ans.* 15.
15. $\frac{12(a + b^2)}{d^3} - (c - b) + \frac{7}{a^2 - b^2} \times \frac{14}{c^2 - d^2}$. *Ans.* $2\frac{2}{3}$.
16. $\left\{\left[\overline{a + b} \times c + d\right] b + a\right\} \times c$. *Ans.* 268.

CHAPTER II.

FUNDAMENTAL OPERATIONS.

ADDITION.

16. ADDITION is the operation of collecting several quantities, and finding the simplest expression for their aggregate. This expression is called the *Sum*.

17. When the quantities are similar, the addition may be performed; when they are not similar, the operation can only be indicated. Thus, the sum of $7a^2b$ and $4a^2b$, is $11a^2b$; in the same way that the sum of 7 pounds and 4 pounds, is 11 pounds. The sum of $3a^2c$ and $4b^2$, cannot be expressed by a single term, any more than the sum of 3 pounds and 4 feet; the sum is indicated, however, by writing the quantities one after the other, with their proper signs; thus,

$$3ac^2 + 4b^2.$$

18. To deduce a rule for adding similar terms, let us take the following examples:

(1.)	(2.)	(3.)	(4.)
$+ 7a^2bc$	$- 2a^2bc$	$+ 4a^2bc$	$- 8a^2bc$
$+ a^2bc$	$- 3a^2bc$	$- 2a^2bc$	$+ 5a^2bc$
$+ 3a^2bc$	$- a^2bc$	$+ 7a^2bc$	$- 2a^2bc$
$+ 5a^2bc$	$- 8a^2bc$	$- 5a^2bc$	$+ 3a^2bc$
$+ 16a^2bc$	$- 14a^2bc$	$+ 4a^2bc$	$- 2a^2bc$

In the first example, the quantity a^2bc is taken positively, $7 + 1 + 3 + 5$, or 16 times, and the sum is $+ 16a^2bc$. In the second example, the quantity is taken negatively, $2 + 3 + 1 + 8$, or 14 times, and the sum is $- 14a^2bc$. In both cases, the sum is found by adding the coefficients, annexing the common literal part, and prefixing the common sign.

In the third example, the quantity a^2bc is taken positively, $4 + 7$, or 11 times, and negatively, $2 + 5$, or 7 times, and the sum is $11 - 7$, or 4 times a^2bc. In the fourth example, the quantity a^2bc is taken, additively, $5 + 3$, or 8 times, and negatively, $8 + 2$, or 10 times, and the sum is $8 - 10$, or $- 2$ times a^2bc.

In like manner, all other groups of similar quantities may be added. Hence, the following rule for adding similar quantities.

RULE.

Add the coefficients of the positive terms and of the negative terms separately; subtract the lesser sum from the greater, prefixing the sign of the greater; then annex to this result the literal part.

19. In adding polynomials together, which contain several groups of similar quantities, it will be found more convenient to write them down, so that each group of similar quantities may fall in a column by itself.

We may, therefore, write the following rule for the addition of algebraic quantities:

RULE.

I. *Write down the quantities to be added, so that similar terms shall fall in the same column.*

II. *Add up each column separately, and then annex the dissimilar terms with their proper signs.*

EXAMPLES.

(1.)

$$\begin{array}{r} 3a - 3bx \\ 9a - 5bx \\ 5a - 4bx \\ \hline 17a - 12bx \end{array}$$

(2.)

$$\begin{array}{r} c + bx^2 + d \\ 4c - 2bx^2 - 2d \\ 5c + 3bx^2 \\ \hline 10c + 2bx^2 - d \end{array}$$

(3.)

$$\begin{array}{r} 3x^2y - 3y^2x - 4y + z \\ 3x^2y + 7y^2x - 8y \\ 8x^2y - 5y^2x + 5y \\ \hline 14x^2y - y^2x - 7y + z \end{array}$$

(4.)

$$\begin{array}{r} 4a + bc + 5d \\ 2a + 2bc + 3d \\ 3a - 3bc \\ \hline 9a + 8d \end{array}$$

(5.)

$$\begin{array}{r} 4cx^2 + 5dy^2 - 2z^3 + d \\ 3cx^2 - 2dy^2 - 2z^3 - d \\ -2cx^2 - dy^2 + 5z^3 \\ \hline 5cx^2 + 2dy^2 + z^3 \end{array}$$

(6.)

$$\begin{array}{r} 4ab - 4c + 2(a+b) \\ 3ab + 5c + 5(a+b) \\ ab + c + 3(a+b) \\ -2ab + 7c - 4(a+b) \\ -ab - c - 2(a+b) \\ \hline 5ab + 8c + 4(a+b) \end{array}$$

(7.)

$$\begin{array}{r} 12x^2y + 2(a+b)z^2 \\ -11x^2y - (a+b)z^2 \\ 4x^2y + 4(a+b)z^2 \\ -3x^2y + 2(a+b)z^2 \\ x^2y + (a+b)z^2 \\ \hline 3x^2y + 8(a+b)z^2 \end{array}$$

Find the sums of the following polynomials:

1. $a+b+c$, $a+b-c$, $a-b+c$, and $-a+b+c$.
   Ans. $2a+2b+2c$.
2. $2ax+3by$, $3ax+2by$, $7ax+by$, and $8ax+7by$.
   Ans. $20ax+13by$.

3. $2a^2 - 17ab + 3b^2$, $5a^2 + 12ab - 5b^2$, $6ab + 12a^2 - 9b^2$, and $3b^2 + 6ab + 3a^2$. *Ans.* $22a^2 + 7ab - 8b^2$.

4. $x^3 - y^3 + 2xy^2 - 3x^2y$, $2x^3 - 3xy^2 - 5x^2y + 2y^3$, $6x^2y + 6xy^2 - x^3 - y^3$, and $5xy^2 - 2y^3 - 4x^3 + 8x^2y$.
Ans. $-2x^3 + 6x^2y + 10xy^2 - 2y^3$.

5. $2x + 3y - 4z - 10$, $8y - 4x + 7z + 8$, $11z + 5x - 10y - 2$, and $16 + 10x + 12y + 14z$.
Ans. $13x + 13y + 28z + 12$.

6. $3x^3 + 2y^3 + z^3 + 8xyz$, $y^3 + 2x^3 - 3z^3 - 4xyz$, $z^3 + 3x^3 - 2y^3 - 2xyz$, and $zyx + x^3 + y^3 + z^3$.
Ans. $9x^3 + 2y^3 + 3xyz$.

7. $x^4 + 3x^3y + x^2z - 2xv$, $30x^4 - 29x^2z + 18xv - 17x^3y$, $22x^3y - 15x^4 - 32x^2z + 16xv$, and $17x^2z - 12x^4 + 6x^3y - 11xv$.
Ans. $4x^4 + 14x^3y - 43x^2z + 21xv$.

8. $ax - by$, $x - y$, $ax - x$, and $ax + x$.
Ans. $3ax + x - by - y$.

9. $ax + 2bx + 4by - 3ay$, $2ax + bx + 2ay - by$, and $4ax + 3by$. *Ans.* $7ax + 3bx + 6by - ay$.

10. $px + qy + rz - c$, $2px - 2qy + 2c$, $3qy - px + 4c$, and $7px - 8qy - rz - 3c$. *Ans.* $9px - 6qy + 2c$.

11. $ax^2 + a^2x - 2ax$, $x - ax + 2x^2$, $ax^2 - 2x + x^2$, and $-2ax - 2a^2x - 2ax^2$.
Ans. $3x^2 - a^2x - 5ax - x$.

12. $a^2x - ax^2 - x^2$, $ax - x^2 - a^2$, $-2a^2 - 2a^2x - 2ax^2$, and $-3a^2x + 3a^2 + 3ax^2$. *Ans.* $-4a^2x - 2x^2 + ax$.

13. $a - x + 4y - 3z + w$, $z - w - y - 3a - 2x$, and $x + y + z$. *Ans.* $-2a - 2x + 4y - z$.

14. $ax^2y + bxy^2z^2 + cxz^3$, $dxy^2z^2 + cxz^3$, and $2ax^2y + 4bdx$. *Ans.* $3ax^2y + bxy^2z^2 + dxy^2z^2 + 2cxz^3 + 4bdx$.

20. The term *sum*, in Algebra has a more extensive signification than in Arithmetic. In Arithmetic, the sum is always greater than either of the parts; in Algebra, it may be numerically less than either. Thus, the sum of $7a^2b$ and $-4a^2b$, is $3a^2b$. To distinguish between these cases, the sum of two or more algebraic quantities, is called their *algebraic sum*.

SUBTRACTION.

21. SUBTRACTION is the operation of finding from two given quantities a third, which, being added to the second, will produce the first.

The first is called the *Minuend*, the second the *Subtrahend*, and the third the *Remainder*, or the *Difference*.

22. To demonstrate a rule for Subtraction, let the sum of the terms of the minuend be denoted by m, and the sum of the terms of the subtrahend by s; from the definition, the remainder must be such a quantity that, being added to s, will produce m; it is, therefore, $m - s$, which is equivalent to $m + (-s)$. Hence, the remainder is found by changing the sign of the subtrahend, and adding the result to the minuend. Changing the sign of the subtrahend, is the same as changing the signs of all its terms, from $+$ to $-$, or from $-$ to $+$.

In practice, it will be found convenient to write the quantities so that similar terms shall fall in the same column. We may therefore write the following

RULE.

I. *Write down the quantities so that the terms of the subtrahend shall fall under the similar ones of the minuend.*

II. *Conceive the signs of all the terms of the subtrahend to be changed from + to −, or from − to +, and proceed as in Addition.*

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$12ab$	$8a^2bc$	$13a^nb$	$5a^pb^qc$
$6ab$	$4a^2bc$	$9a^nb$	$2a^pb^qc$
$6ab$	$4a^2bc$	$4a^nb$	$3a^pb^qc$

(5.)	(6.)	(7.)	(8.)
$7ac$	$10b^2d$	$-8a^2bc$	$-3a^4b^n$
$-4ac$	$-3b^2d$	$+3a^2bc$	$-5a^4b^n$
$11ac$	$13b^2d$	$-11a^2bc$	$2a^4b^n$

(9.)	(10.)	(11.)
$6a^2-8b$	$3x^2-4x^2y+8$	$4xy^2+4z$
$3a^2-5b$	$5x^2-6x^2y-3$	$-3xy^2+7z-6x^3$
$3a^2-3b$	$-2x^2+2x^2y+11$	$7xy^2-3z+6x^3$

12. From $2a+b-c$, subtract $a-b$.

Ans. $a+2b-c$.

13. From $3ac-2b$, subtract $ac-b-d$.

Ans. $2ac-b+d$.

14. From $5ab-6$, subtract $-2ab+6$.

Ans. $7ab-12$.

15. From $4y^2 - 3y + 4$, subtract $2y^2 + 2y + 4$.

Ans. $2y^2 - 5y$.

16. From $219a^3 - 117a^2b + 218ab^2 + 145b^3$ subtract $26a^3 + 4a^2b + 61ab^2 - 10b^3$.

Ans. $193a^3 + 157ab^2 - 121a^2b + 155b^3$.

17. From $a - x + 2y - 3z + w$, subtract $2x + 3a - y + z - w$. *Ans.* $-2a - 3x + 3y - 4z + 2w$.

18. From $5x^3 + x^2y - 6xy^2 + y^3$, subtract $3x^3 + 4x^2y - 7xy^2 + y^3 - xy^3$. *Ans.* $2x^3 - 3x^2y + xy^2 + xy^3$.

19. From $y^4 - 4xy^3 + 7x^2y^2 - x^3y + 3x^4$, subtract $2x^4 + 3x^3y + x^2y^2 + xy^3$.

Ans. $x^4 - 4x^3y + 6x^2y^2 - 5xy^3 + y^4$.

20. From $2px^2 + ry^2 - 3qxy$, subtract $px^2 + qxy - ry^2$.

Ans. $px^2 - 4qxy + 2ry^2$.

21. From $2x^3 - 3x^2y + 2xy^2$, subtract $x^3 - xy^2 + y^3$.

Ans. $x^3 + 3xy^2 - 3x^2y - y^3$.

22. From $7x^2 - xyz + 18z$, subtract $-3x^2 - 2xyz - p - q^3$.

Ans. $10x^2 + xyz + 18z + p + q^3$.

23. The term *difference*, in Algebra, has a more extensive signification than in Arithmetic. In Arithmetic, the difference is always less than the minuend; whereas, in Algebra, the difference may be greater than the minuend. Thus, the difference of $7a^2b$ and $-4a^2b$, is $11a^2b$. To distinguish between these two cases, the difference of two algebraic quantities is called their *algebraic difference*.

24. To indicate that one polynomial is to be subtracted from another, we enclose it in a parenthesis, and write it

after the other, prefixing the sign $-$; thus, the expression,

$$a^3 + b^2x - y^3 - (bc^2 - ad^3 + y),$$

indicates that the polynomial $bc^2 - ad^3 + y$ is to be taken from the polynomial $a^3 + b^2x - y^3$. Performing the operation indicated, by the rule for subtraction, we have the equivalent expression,

$$a^3 + b^2x + y^3 - bc^2 + ad^3 - y.$$

The latter expression may be changed to the former, by changing the signs of the last three terms, inclosing them in a parenthesis, and prefixing the sign $-$. In like manner, any other polynomial may be transformed, as indicated in the following example:

$$\begin{aligned}
&4x^4 - 14x^3y + 4x^2z - 2w + 10,\\
&4x^4 - (14x^3y - 4x^2z + 2w - 10),\\
&4x^4 - 14x^3y - (-4x^2z + 2w - 10),\\
&4x^4 - 14x^3y + 4x^2z - (2w - 10).
\end{aligned}$$

These expressions are all equivalent, the first form being the simplest.

EXAMPLES.

Reduce the following expressions to their simplest form:

1. $2x^3 - 3x^2y + 2xy^2 - (x^3 + y^3 - xy^2)$.

   Ans. $x^3 - 3x^2y + 3xy^2 - y^3$.

2. $3x - \{x - 3a - (2y - a)\}$.

   Ans. $2x + 2y + 2a$.

3. $a^2 - (b^2 - c^2) - \{b^2 - (c^2 - a^2)\} + c^2 - (b^2 - a^2)$.

   Ans. $a^2 - 3b^2 + 3c^2$.

4. $x + y + z - (x - y) - (y - z) - (-y)$.

   Ans. $2y + 2z$.

25. From the preceding principles, we have

$$a - (+b) \text{ equal to } a - b; \text{ and}$$
$$a + (-b) \text{ equal to } a + b.$$

The sign immediately preceding b, is called the *Sign of the Quantity;* the sign preceding the parenthesis, is called the *Sign of Operation;* the sign resulting from the operation, is called the *Essential Sign.* It is plain, that when the sign of the quantity is the same as the sign of operation, the essential sign will be $+$; when the sign of the quantity is contrary to the sign of operation, the essential sign will be $-$.

MULTIPLICATION.

26. MULTIPLICATION is the operation of finding the product of two quantities.

The quantity to be multiplied, is called the *Multiplicand;* that by which it is to be multiplied, is called the *Multiplier;* both are called *factors* of the product.

27. To deduce a rule for multiplying one monomial by another, take the two monomials, $3a^2b$ and $4abc$. The product is indicated thus,

$$3a^2b \times 4abc.$$

But it has been shown in Arithmetic, that the order of the factors may be changed without affecting the value of the product. The factors of the same kind may therefore be written together, giving,

$$3 \times 4aaabbc;$$

or, simplifying the form of the expression, we have,

$$3a^2b \times 4abc = 12a^3b^2c.$$

In like manner, the product of any two monomials may be found. Hence, the

RULE.

Multiply the coefficients together for a new coefficient; write after this, all the letters in the two monomials, giving to each letter an exponent equal to the sum of its exponents in the two factors.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$3abc$	$7a^3b^2c$	$3wx^4y^5z$	$8ax^4y^3z$
$4xy$	$3a^2bc$	$2xyz^6$	$3a^3xyz^3$
$12abcxy$	$21a^5b^3c^2$	$6wx^5y^4z^7$	$24a^4x^5y^4z^4$

5. Multiply $7abc$ by $5ac$. *Ans.* $35a^2bc^2$.
6. Multiply $3ax$ by $7ac$. *Ans.* $21a^2cx$.
7. Multiply $2ay$ by $3a^2xy$. *Ans.* $6a^3xy^2$.
8. Multiply $2a^3y^2x$ by $3ax^3$. *Ans.* $6a^4y^2x^4$.
9. Multiply $12a^3x$ by $4a^3y$. *Ans.* $48a^6xy$.
10. Multiply $12a^n$ by $3a^m$. *Ans.* $36a^{m+n}$.
11. Multiply $7a^nx^m$ by a^nx. *Ans.* $7a^{2n}x^{m+1}$.
12. Find the product of $8abc$, $7a^3bx$, and $3abx$.
Ans. $168a^5b^3cx^2$.
13. Find the product of $3pqr$, $2p^2qr^3$, and $4pr^4x$.
Ans. $24p^4q^2r^8x$.
14. Find the product of $3m^pd$, $4md^2$, and $5mnd$.
Ans. $60m^{p+2}nd^4$.

28. To deduce a rule for multiplying one polynomial with another. First, let it be required to multiply $a - b$ by c. If we multiply a by c, the product ac will be greater than the required product, by the product of b by c, or bc, because b ought to have been taken from a before the multiplication was performed. Hence, the true product is $ac - bc$, and the operation is performed as shown below:

$$\begin{array}{r} a - b \\ c \\ \hline ac - bc. \end{array}$$

Now, let it be required to multiply $a - b$ by $c - d$. Multiplying each term of $a - b$ by $(c - d)$, as above, we have

$$\begin{array}{r} a \quad - b \\ (c - d) \\ \hline a(c - d) - b(c - d) \\ = ac - ad - (bc - bd) \\ = ac - ad - bc + bd. \end{array}$$

The same result may be obtained by multiplying each term of the multiplicand by each term of the multiplier, and giving to the product of those terms having like signs the sign $+$, and to the product of the terms having unlike signs, the sign $-$, as indicated below:

$$\begin{array}{r} a - b \\ c - d \\ \hline ac - bc \\ \quad - ad + bd \\ \hline ac - bc - ad + bd. \end{array}$$

Since any two polynomials may be multiplied in the same way, we have the following

RULE.

Multiply every term of the multiplicand by each term of the multiplier, observing that like signs give $+$, *and unlike signs* $-$; *then reduce the result to the simplest form.*

EXAMPLES.

(1.)

$$\begin{array}{l} x^2 + 2ax + a^2 \\ x + a \\ \hline x^3 + 2ax^2 + a^2x \\ \qquad ax^2 + 2a^2x + a^3 \\ \hline x^3 + 3ax^2 + 3a^2x + a^3. \end{array}$$

(2.)

$$\begin{array}{l} 2a^4b + a^3b^2 \\ 2a^2b - 2ab^2 \\ \hline 4a^6b^2 + 2a^5b^3 \\ \qquad - 4a^5b^3 - 2a^4b^4 \\ \hline 4a^6b^2 - 2a^5b^3 - 2a^4b^4. \end{array}$$

It will be found convenient to *arrange* the terms of the polynomials with reference to some letter; that is, to write them down, so that the highest power of that letter shall enter the first term; the next highest, the second term, and so on to the last term. The letter with reference to which the arrangement is made, is called the *leading letter.* In the first of the above examples, the leading letter is x; in the second, it is a. The leading letter of the product will be the same as that of the factors.

EXAMPLES.

1. Multiply $x^2 - xy + y^2$ by $x + y$.
   Ans. $x^3 + y^3$.
2. Multiply $x^2 - xy + y^2$ by $x^2 + xy + y^2$.
   Ans. $x^4 + x^2y^2 + y^4$.
3. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 6$.
   Ans. $3x^4 + 4x^3y - 4x^2y^2 - 13x^2 + 22xy - 30$.

4. Multiply $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$ by $x + y$. *Ans.* $x^7 + y^7$.

5. Multiply $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$ by $x + 2y$. *Ans.* $x^5 + 32y^5$.

6. Multiply $27a^2 - 13ab + 5b^2$ by $7a^2 + b^2$. *Ans.* $189a^4 - 91a^3b + 62a^2b^2 - 13ab^3 + 5b^4$.

7. Multiply together $a + x$, $b + x$, and $c + x$. *Ans.* $x^3 + ax^2 + bx^2 + cx^2 + abx + acx + bcx + abc$.

8. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$. *Ans.* $a^3 + b^3 + c^3 - 3abc$.

9. Multiply $a^4 + a^3b + a^2b^2 + ab^3 + b^4$ by $a - b$. *Ans.* $a^5 - b^5$.

10. Multiply $2a + bc - 2b^2$ by $2a - bc + 2b^2$. *Ans.* $4a^2 - b^2c^2 + 4b^3c - 4b^4$.

11. Multiply $4ab - 2ac$ by $6ab + 3ac$. *Ans.* $24a^2b^2 - 6a^2c^2$.

12. Multiply $a + bx$ by $a + cx$. *Ans.* $a^2 + abx + acx + bcx^2$.

13. Multiply together $x - 10$, $x + 1$, and $x + 4$. *Ans.* $x^3 - 5x^2 - 46x - 40$.

14. Multiply together $x - 5$, $x - 6$, $x - 7$ and $x + 8$. *Ans.* $x^4 - 10x^3 - 37x^2 + 646x - 1680$.

15. Multiply $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^3 - 3a^2b + 3ab^2 - b^3$. *Ans.* $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$.

16. Multiply together $x^2 - a^2$, $x^2 - ax + a^2$, and $x^2 + ax + a^2$. *Ans.* $x^6 - a^6$.

17. Multiply $x^n - 2ax^{n-1}y + y^q$ by $3xy^n$. *Ans.* $3x^{n+1}y^n - 6ax^ny^{n+1} + 3xy^{n+q}$.

18. Multiply $a^n + 2a^m b^n + ab^p$ by $a^m - b^n$.

Ans. $a^{m+n} + 2a^{2m}b^n + a^{m+1}b^p - a^n b^n - 2a^m b^{2n} - ab^{p+n}$.

19. Multiply $x^n + y^n$ by $x^n + y^n$.

Ans. $x^{2n} + 2x^n y^n + y^{2n}$

20. Multiply together $x^m - y^m$, $x^m + y^m$, and $x^n - y^n$.

Ans. $x^{2m+n} - x^n y^{2m} - x^{2m} y^n + y^{2m+n}$.

DIVISION.

29. Division is the operation of finding from two quantities a third, which being multiplied by the second, will produce the first.

The first is called the *Dividend*, the second the *Divisor*, and the third the *Quotient*.

Division is the converse of Multiplication. In it we have given the product and one factor, to find the other. The rules for Division are just the converse of those for Multiplication.

30. To divide one monomial by another. Take the example,

$$\frac{56a^5b^3c^2d}{7a^4bc};$$

The quotient must be such a monomial as, being multiplied by the divisor, will give the dividend. Its coefficient then must be such that, being multiplied by 7, will give 56, and the exponent of each letter which enters it must be such that, when added to the exponent of the same letter in the divisor, will give the exponent of the same letter in the dividend. Hence, the following rule for dividing one monomial by another.

RULE.

Divide the coefficient of the dividend by that of the divisor for a new coefficient; after this, write all the letters in the dividend, giving to each an exponent equal to the excess of its exponent in the dividend over that in the divisor.

With respect to signs, it follows from the rule for signs in Multiplication, that the quotient of terms having like signs must be $+$, and that of terms having unlike signs, must be $-$.

EXAMPLES.

(1.) $\dfrac{+16a^3b^3c^2d}{+4ab^2cd} = +4a^2bc.$

(2.) $\dfrac{-18a^2b^2x^5y^4z}{-6abx^3y^2z} = 3abx^2y^4.$

(3.) $\dfrac{+15a^3x^2y^4z^3}{-5a^2xy^3z^2} = -3axyz.$

(4.) $\dfrac{-21p^3y^3x^2z}{+7p^3yx} = -3y^2xz.$

5. Divide $21fg^2h$ by $7fg$. *Ans.* $3gh$.
6. Divide $84a^2x^3y$ by $-12axy$. *Ans.* $-7ax^2$.
7. Divide $-36a^2b^3c$ by $18ab$. *Ans.* $-2ab^2c$.
8. Divide $-25a^2x^2y^4$ by $-5x^2y^2$. *Ans.* $5a^2y^2$.
9. Divide $72a^4b^4c^2$ by $-36a^2b^2$. *Ans.* $-2a^2b^2c^2$.
10. Divide $-14a^2x^3y^4$ by $-7ax^2y^3$. *Ans.* $2axy$.
11. Reduce the expression $\dfrac{14ax^3z}{7axz}$, to its simplest form. *Ans.* $2x^2$.
12. Reduce $\dfrac{-24fg^2h}{6g}$. *Ans.* $-4fgh$.

13. Reduce $\left(\frac{a^6b^6c^6}{a^3b^3c}\right) \div ab^2c^3$. *Ans.* a^2bc^2.

14. Reduce $(300a^3c^3 \div 10ac) \div 6a^2c$. *Ans.* $5c$.

31. In dividing monomials, it often happens that the exponents of the same letter are equal, in which case, that letter disappears in the quotient. It might, however, be retained by giving to it the exponent 0. Thus,

$$\frac{a^m}{a^m} = a^{m-m} = a^0; \text{ but } \frac{a^m}{a^m} = 1; \therefore a^0 = 1.$$

From this we infer, that, *the* 0 *power of any quantity is equal to* 1, *and that* 1 *is equal to the* 0 *power of any quantity. Any quantity may therefore be introduced into a term, as a factor, by giving it the exponent* 0.

32. When the exponent of any letter is less in the dividend than it is in the divisor, the exponent of that letter in the quotient will be negative. The negative exponent may, however, be retained, and it can be shown that it is subject to all the rules of operation deduced for positive exponents. To explain the meaning of negative exponents, we have,

$$\frac{a^2}{a^5} = a^{-3}; \text{ also, } \frac{a^2}{a^5} = \frac{1}{a^3}; \therefore a^{-3} = \frac{1}{a^3}.$$

Hence, we infer, that, *any quantity having a negative exponent, is equal to the reciprocal of the same quantity with an equal positive exponent.* We also infer, that, *any factor may be transferred from the denominator to the numerator of a fraction, or the reverse, by changing the sign of its exponent.*

EXAMPLES.

(1.) $\frac{7a^3b^2c}{4a^2b^2} = \frac{7}{4}ab^0c = \frac{7}{4}ac.$ (2.) $\frac{6ab^3c}{3a^2b^4c^2} = 2a^{-1}b^{-1}c^{-1} = \frac{2}{abc}.$

3. Reduce $\frac{15ab^2c}{5a^2b^3}$. *Ans.* $3a^{-1}b^{-1}c$, or $\frac{3c}{ab}$.

4. Reduce $\frac{17xyz}{5x^3y^2z}$. *Ans.* $\frac{17}{5}x^{-2}y^{-1}$, or $\frac{17}{5x^2y}$.

5. Reduce $\frac{24fgx}{6fg^2x^3}$. *Ans.* $4f^0g^{-1}x^{-2}$, or $\frac{4}{gx^2}$.

6. Reduce $\frac{-8a^{-1}x^{-2}}{4ax^2}$. *Ans.* $-2a^{-2}x^{-4}$, or $-\frac{2}{a^2x^4}$.

7. Reduce $\frac{-6a^{-3}x^{-4}}{-3a^{-4}x^{-5}}$. *Ans.* $2ax$.

8. Reduce $72a^2b^3 \div 12a^3$. *Ans.* $6a^{-1}b^3$, or $\frac{6b^3}{a}$.

9. Reduce $\left(\frac{-36ax^4}{12a^2x}\right) \div 3ax^5$.

Ans. $-a^{-2}x^{-2}$, or $-\frac{1}{a^2x^2}$.

10. Reduce $(14ax \div 7yz) \div 2xz$.

Ans. $ay^{-1}z^{-2}$, or $\frac{a}{yz^2}$.

33. To divide a polynomial by a monomial: *Divide each term separately.*

EXAMPLES.

(1.) $\frac{6ab-8ax+4a^2y}{2a} = 3b-4x+2ay.$ (2.) $\frac{10a^2x-15x^2}{5x} = 2a^2-3x$

3. Divide $5xy + 20x^2y - 45axy$ by $5xy$.

Ans. $1 + 4x - 9a$.

4. Divide $-9a^2bc - 12ab^2c + 15abc^2$ by $-3abc$.

Ans. $3a + 4b - 5c$.

5. Divide $14a^2xy^3 - 7aby^4 - 13y^4$ by $7a^3x^3y^3$.

Ans. $2a^{-1}x^{-2} - a^{-2}bx^{-3}y - \frac{13}{7}a^{-3}x^{-3}y$.

6. Divide $27a^m - 18a^{2m}b^n - 21a^p$ by $3a^m$.

Ans. $9 - 6a^mb^n - 7a^{p-m}$.

7. Divide $12a^4(a+x)^2 - 18a^3(a+x)^3 + 24a^2(a+x)^4$ by $6a^2(a+x)^2$. *Ans.* $2a^2 - 3a(a+x) + 4(a+x)^2$.

34. To divide one polynomial by another. Take the example:

Dividend.	Divisor.
$9x^3 + 12x^2 + 16x + 8$	$3x + 2$
$9x^3 + 6x^2$	$3x^2 + 2x + 4$; Quotient.
$6x^2 + 16x$	
$6x^2 + 4x$	
$12x + 8$	
$12x + 8$	
0	Remainder.

The dividend and divisor are both arranged with reference to the same letter; then will the quotient of the first term of the dividend, by the first term of the divisor, be the first term of the quotient. The product of the divisor by this term, being subtracted from the dividend, gives a new dividend, which may be treated in the same way as the original one, and so on to the end of the process. For convenience of multiplication, the divisor is written on the right of the dividend, and the quotient under it.

In all other respects, the operation is entirely similar to division in Arithmetic. Since all other similar cases may be treated in the same way, we have the following

RULE.

I. *Arrange both polynomials with reference to the same letter.*

II. *Divide the first term of the dividend by the first term of the divisor, for the first term of the quotient. Multiply the divisor by this term of the quotient, and subtract the product from the dividend.*

III. *Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient. Multiply the divisor by this term, and subtract the product from the first remainder, and so on.*

IV. *Continue the operation, until a remainder is found equal to* 0, *or one whose first term is not divisible by that of the divisor.*

When a remainder is found equal to 0, the division is exact. When a remainder is found whose first term is is not divisible by that of the divisor, without giving rise to fractions, the exact division is impossible. In that case, write the last remainder after the quotient found, placing under it the divisor, in the form of a fraction.

EXAMPLES.

(1.)

$$\begin{array}{r|l} 6a^3x^3 + 13a^2x^2 + 6ax & 2a^2x^2 + 3ax \\ \underline{6a^3x^3 + 9a^2x^2} & \hline 3ax + 2 \\ 4a^2x^2 + 6ax & \\ \underline{4a^2x^2 + 6ax} & \\ 0 & \end{array}$$

(2.)

$$\begin{array}{l|l} a^4 + a^3x + a^2x + ax^2 - 2x & a + x \\ \underline{a^4 + a^3x} & \overline{a^3 + ax} - \dfrac{2x}{x+a} \\ \quad + a^2x + ax^2 & \\ \quad \underline{+ a^2x + ax^2} & \\ \qquad\qquad - 2x & \end{array}$$

Here the quotient is fractional, and the division is not exact

(3.)

$$\begin{array}{l|l} 1 + x & 1 - x \\ \underline{1 - x} & \overline{1 + 2x + 2x^2} + 2x^3 +, \text{ \&c.} \\ \quad + 2x & \\ \quad \underline{+ 2x - 2x^2} & \\ \qquad + 2x^2 & \\ \qquad \underline{+ 2x^2 - 2x^3} & \\ \qquad\qquad + 2x^3 & \end{array}$$

In this example, the operation does not terminate, but may be continued to any desired extent.

EXAMPLES.

1. Divide $a^2 + 4ax + 4x^2$ by $a + 2x$. *Ans.* $a + 2x$.

2. Divide $a^3 - 3a^2x + 3ax^2 - x^3$ by $a - x$.
Ans. $a^2 - 2ax + x^2$.

3. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.
Ans. $a^2 + 4ax + x^2$.

4. Divide $a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4$ by $a^2 - 2ay + y^2$. *Ans.* $a^2 - 2ay + y^2$.

5. Divide $a^4 - b^4$ by $a^3 + a^2b + ab^2 + b^3$. *Ans.* $a - b$.

6. Divide $12x^4 - 192$ by $3x - 6$.

Ans. $4x^3 + 8x^2 + 16x + 32$.

7. Divide $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ by $x^3 - 3x^2y + 3xy^2 - y^3$. *Ans.* $x^3 + 3x^2y + 3xy^2 + y^3$.

8. Divide $x^{4n} + x^{2n}y^{2n} + y^{4n}$ by $x^{2n} + x^ny^n + y^{2n}$.

Ans. $x^{2n} - x^ny^n + y^{2n}$.

9. Divide $a^2 - b^2 + 2bc - c^2$ by $a - b + c$.

Ans. $a + b - c$.

10. Divide $x^4 - 6x^2y^2 - 16xy^3 - 15y^4$ by $x^2 + 2xy + 3y^2$. *Ans.* $x^2 - 2xy - 5y^2$.

11. Divide $ax^3 - a^2x^2 - bx^2 + b^2$ by $ax - b$.

Ans. $x^2 - ax - b$.

12. Divide $mpx^3 + mqx^2 - npx^2 - mrx - nqx + nr$ by $mx - n$. *Ans.* $px^2 + qx - r$.

13. Divide $a^3x^2 - a^3x + a^2x^2 + 2a^2x - 2a^2 + 2ax + ax^2 - ax^3 - x^3$ by $a^2x + 2a - x^2$. *Ans.* $ax - a + x$.

14. Divide $-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8$ by $2a^{-3}x^3 - 3ax^4$. *Ans.* $-a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4$.

15. Divide $a^3 - 3a^2x + x^3$ by $a + x$.

Ans. $a^2 - 4ax + 4x^2 - \dfrac{3x^3}{a+x}$.

16. Divide $a^5 + a^3b^2 + 2a^2b^3 - b^5$ by $a^2 - ab + b^2$.

Ans. $a^3 + a^2b + ab^2 + 2b^3 + \dfrac{ab^4 - 3b^5}{a^2 - ab + b^2}$.

17. Divide $x^3 + ax^2 + bx + c$ by $x - r$.

Ans. $x^2 + rx + ax + r^2 + ar + b + \dfrac{r^3 + ar^2 + br + c}{x - r}$.

18. Divide $1 + 2x$ by $1 - 3x$.

Ans. $1 + 5x + 15x^2 + 45x^3 +$, &c.

19. Divide $1 + 2x$ by $1 - x - x^2$.

Ans. $1 + 3x + 4x^2 + 7x^3 +$, &c.

20. Divide 1 by $1 + x$.

Ans. $1 - x + x^2 - x^3 + x^4$, &c.

USEFUL FORMULAS.

35. A FORMULA, is an algebraic expression of a general rule, or principle.

The verification of the following formulas, affords additional exercises in Multiplication.

1. $(x + y)^2 = (x + y)(x + y) = x^2 + 2xy + y^2$.
2. $(x - y)^2 = (x - y)(x - y) = x^2 - 2xy + y^2$.
3. $(x + y)(x - y) = x^2 - y^2$.

36. Formulas serve to shorten algebraic operations, and are also of much use in the operation of factoring. When translated into common language, they give rise to practical rules. These three may be translated as follows, since x and y are any quantities whatever:

1. *The square of the sum of any two quantities, is equal to the square of the first, plus twice the product of first and second, plus the square of the second.*

2. *The square of the difference of any two quantities, is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

3. *The product of the sum and difference of two quantities, is equal to the square of the first, minus the square of the second.*

EXAMPLES.

1. Let it be required to find the square of $2a + 3x$. The square of $2a$, is $4a^2$; twice the product of $2a$ and $3x$, is $12ax$; the square of $3x$, is $9x^2$. Hence, by the first rule,

$$(2a + 3x)^2 = 4a^2 + 12ax + 9x^2.$$

2. Find the square of $2a - 3x$. By the second rule, we have, as before,

$$(2a - 3x)^2 = 4a^2 - 12ax + 9x^2.$$

3. Find the product of $2a + 3x$, and $2a - 3x$. By the third rule, we have, as before,

$$(2a + 3x)(2a - 3x) = 4a^2 - 9x^2.$$

In like manner, solve the following examples:

4. Find the square of $ax + by$.

Ans. $a^2x^2 + 2abxy + b^2y^2$.

5. Find the square of $7x^2 + 3y^2$.

Ans. $49x^4 + 42x^2y^2 + 9y^4$.

6. Find the square of $8ab + 4cd$.

Ans. $64a^2b^2 + 64abcd + 16c^2d^2$.

7. Find the square of $2ac - 3d$.

Ans. $4a^2c^2 - 12acd + 9d^2$.

8. Find the square of $16xy - 7y^2$.

Ans. $256x^2y^2 - 224xy^3 + 49y^4$.

9. Find the square of $2ab - cd$.

Ans. $4a^2b^2 - 4abcd + c^2d^2$.

10. Find the product of $2a + 3x$, and $2a - 3x$.

Ans. $4a^2 - 9x^2$.

11. Find the product of $7b + 4c$, and $7b - 4c$.

Ans. $49b^2 - 16c^2$.

12. Find the product of $8xy + 3x^2$, and $8xy - 3x^2$.

Ans. $64x^2y^2 - 9x^4$.

By reversing these operations, the squares and products above found may be resolved into binomial factors.

37. Let the student verify the following additional formulas, by actual multiplication and division.

4. $(x + a)(x + b) = x^2 + (a + b)x + ab$.
5. $(x^2 + xy + y^2)(x - y) = x^3 - y^3$.
6. $(x^2 - xy + y^2)(x + y) = x^3 + y^3$.
7. $(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + x^2y^2 + y^4$.
8. $(x + y)(x - y)(x^2 + y^2) = x^4 - y^4$.
9. $\dfrac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 +$, &c., $+ y^{n-1}$.
10. $\dfrac{x^{mn} - y^{mn}}{x^n - y^n} = x^{mn-n} + x^{mn-2n}y^n + x^{mn-3n}y^{2n} +$, &c., $+ y^{mn-n}$.

The general demonstration of Formulas (9) and (10), will be found in the Appendix.

FACTORING.

37. FACTORING is the operation of resolving a quantity into factors. The principles employed are the converse of those used in Multiplication. The operation of factoring must be performed by inspection.

When every term of a polynomial contains the same monomial factor, that factor is a factor of the polynomial, and the other factor may be found by dividing the polynomial by the monomial factor.

EXAMPLES.

1. Factor the polynomial $8a^2x^2 + 4a^3x$.

Here, we see that $4a^2x$ is a factor common to each term, hence it is one of the required factors. Dividing by $4a^2x$, we have the quotient, $2x + a$, which is the other factor; or,

$$8a^2x^2 + 4a^3x = 4a^2x(2x + a).$$

In like manner, the following polynomials may be factored.

2. Factor $7a^2bc^3 - 28abc$. *Ans.* $7abc(ac^2 - 4)$.

3. Factor $4x^4y^2 - 2x^2y^2$. *Ans.* $2x^2y^2(2x^2 - 1)$.

4. Factor $6x^2y^2 + 12xy^3$. *Ans.* $6xy^2(x + 2y)$.

5. Factor $2a^2b + abc - abd$. *Ans.* $ab(2a + c - d)$.

6. Factor $7x^3y^2 - 7x^2y^3 + 7x^2y^2z$.
Ans. $7x^2y^2(x - y + z)$.

7. Factor $15a^2cd + 20ac^2d - 15acd^2$.
Ans. $5acd(3a + 4c - 3d)$.

38. When two terms of a trinomial are squares, and the third term is equal to twice the product of their square roots, Formula (1) enables us to resolve the trinomial into two factors, as in the following examples:

8. Factor $a^2 + 2ab + b^2$. *Ans.* $(a + b)(a + b)$.

9. Factor $4x^2 + 12xy + 9y^2$.
Ans. $(2x + 3y)(2x + 3y)$.

10. Factor $x^2 + 12x + 36$. *Ans.* $(x + 6)(x + 6)$.

11. Factor $4x^4 + 4x^2y + y^2$.
Ans. $(2x^2 + y)(2x^2 + y)$.

12. Factor $4a^2b^2 + 12abc + 9c^2$.
Ans. $(2ab + 3c)(2ab + 3c)$.

13. Factor $16a^4y^4 + 8a^2y^4z^2 + y^4z^4$.
Ans. $(4a^2y^2 + y^2z^2)(4a^2y^2 + y^2z^2)$.

39. When two terms of a trinomial are squares, and the third term is equal to *minus* twice the product of their square roots, the trinomial may be factored by means of Formula (2), as in the following examples:

14. Factor $a^2 - 2ab + b^2$. *Ans.* $(a - b)(a - b)$.

15. Factor $a^2x^2 - 2acx + c^2$. *Ans.* $(ax - c)(ax - c)$.

16. Factor $4x^2 - 4xy + y^2$. *Ans.* $(2x - y)(2x - y)$.

17. Factor $9a^2b^2 - 24a^2bc + 16a^2c^2$.
Ans. $(3ab - 4ac)(3ab - 4ac)$.

18. Factor $4x^4 - 4x^2y + y^2$. *Ans.* $(2x^2 - y)(2x^2 - y)$.

19. Factor $36x^2 - 24xy + 4y^2$.
Ans. $(6x - 2y)(6x - 2y)$.

20. Factor $4x^2y^2 - 4xyz + z^2$.
Ans. $(2xy - z)(2xy - z)$.

40. When the two terms of a binomial are squares, and have contrary signs, the binomial may be factored by means of Formula (3), as in the following examples:

21. Factor $a^2 - b^2$. *Ans.* $(a + b)(a - b)$.

22. Factor $4x^2 - 9y^2$. *Ans.* $(2x + 3y)(2x - 3y)$.

23. Factor $a^2c^2 - b^2d^2$. *Ans.* $(ac + bd)(ac - bd)$.

24. Factor $9a^2x^2 - 16a^2y^2$.

Ans. $(3ax + 4ay)(3ax - 4ay)$.

25. Factor $25a^4b^4x^4 - 4z^2$.

Ans. $(5a^2b^2x^2 + 2z)(5a^2b^2x^2 - 2z)$.

26. Factor $49x^4 - 16y^2$. *Ans.* $(7x^2 + 4y)(7x^2 - 4y)$.

In like manner, we have, from Formula (4):

27. $x^2 + 13x + 42 = x^2 + (6 + 7)x + 6 \times 7 = (x + 6)(x + 7)$.

28. $x^2 + 2x - 15 = x^2 + (5 - 3)x - 3 \times 5 = (x - 3)(x + 5)$.

29. $x^2 - 15x + 56 = x^2 - (7 + 8)x - 7 \times -8 = (x - 7)(x - 8)$.

30. $x^2 - x - 72 = x^2 + (8 - 9)x - 9 \times 8 = (x + 8)(x - 9)$.

In like manner, we have, from Formulas (5), (6), (7), and (8):

31. $8a^3 - b^3 = (2a - b)(4a^2 + 2ab + b^2)$.

32. $a^3 + 64m^3 = (a + 4m)(a^2 - 4am + 16m^2)$.

33. $16a^4 + 36a^2b^2 + 81b^4 = (4a^2 + 6ab + 9b^2)(4a^2 - 6ab + 9b^2)$.

34. $a^4b^4 - 81c^4 = (a^2b^2 + 9c^2)(a^2b^2 - 9c^2) = (a^2b^2 + 9c^2)(ab + 3c)(ab - 3c)$.

MISCELLANEOUS EXAMPLES.

Factor the following expressions:

35. $9x^4y^2 + 24x^3y^3 + 16x^2y^4$.

Ans. $(3x^2y + 4xy^2)\,(3x^2y + 4xy^2)$.

36. $4x^2 - 12xy + 9y^2$. *Ans.* $(2x - 3y)\,(2x - 3y)$.

37. $a^2b^2c^2 - c^2d^2$. *Ans.* $c^2(ab + d)\,(ab - d)$.

38. $x^2 + 9x + 18$. *Ans.* $(x + 6)\,(x + 3)$.

39. $2a^2x^2 - 2b^2x^2$. *Ans.* $2x^2(a + b)\,(a - b)$.

40. $a^2 - b^2 + 2bc - c^2$.

Ans. $a^2 - (b - c)^2 = (a + b - c)\,(a + c - b)$.

41. $a^4 - 9a^2b^2 - 6abc^2 - c^4$.

Ans. $a^4 - (3ab + c^2)^2 = (a^2 + 3ab + c^2)\,(a^2 - 3ab - c^2)$.

CHAPTER III.

GREATEST COMMON DIVISOR, AND LEAST COMMON MULTIPLE.

GREATEST COMMON DIVISOR.

41. A COMMON DIVISOR of two quantities, is a quantity that will divide them both without a remainder. Thus, $3a^2b$, is a common divisor of $9a^2b^2c$ and $3a^2b^2 - 6a^3b^3$.

42. A SIMPLE or PRIME FACTOR is one that cannot be resolved into any other factors.

Every prime factor, common to two quantities, is a common divisor of those quantities. The continued product of any number of prime factors, common to two quantities, is also a common divisor of those quantities.

43. The GREATEST COMMON DIVISOR of two quantities, is the continued product of all the prime factors which are common to both.

44. When both quantities can be resolved into prime factors, by the method of factoring already given, the Greatest Common Divisor may be found by the following

RULE.

I. *Resolve both quantities into their prime factors.*

II. *Find the continued product of all those factors which are common to both; it will be the Greatest Common Divisor required.*

EXAMPLES.

1. Let it be required to find the greatest common divisor of $42abx$ and $70acx$.

Factoring, we have,

$$42abx = 7a \times 2x \times 3b,$$
$$70acx = 7a \times 2x \times 5c.$$

The factors $7a$ and $2x$ are common: hence, the divisor sought is $7a \times 2x$, or $14ax$.

2. Find the greatest common divisor of $3ax^2 + 3x^3$ and $2ay + 2xy$.

Factoring, we have,

$$3ax^2 + 3x^3 = 3x^2(a + x),$$
$$2ay + 2xy = 2y\,(a + x):$$

hence, the divisor sought is $a + x$.

3. Find the greatest common divisor of $2a^3 - 4a^2b + 2ab^2$ and $2a^3 - 2ab^2$.

Factoring, we have,

$$2a^3 - 4a^2b + 2ab^2 = 2a(a^2 - 2ab + b^2) = 2a(a-b)\,(a-b),$$
$$2a^3 - 2ab^2 = 2a(a^2 - b^2) = 2a(a-b)\,(a+b):$$

hence, the divisor sought is $2a(a - b)$, or $2a^2 - 2ab$.

4. Find the greatest common divisor of $56acdx^2y$ and $24afx^2y$. *Ans.* $8ax^2y$.

5. Find the greatest common divisor of $4a^2c - 4acx$ and $3a^2g - 3agx$. *Ans.* $a(a - x)$, or $a^2 - ax$.

6. Find the greatest common divisor of $x^3 - y^3$ and $x^2 - y^2$. *Ans.* $x - y$.

7. Find the greatest common divisor of $4c^2 - 12cx + 9x^2$ and $4c^2 - 9x^2$. *Ans.* $2c - 3x$.

8. Find the greatest common divisor of $4ax^3 - 4axy^2$ and $12a^2x^2 - 12a^2y^2$. *Ans.* $4ax^2 - 4ay^2$.

9. Find the greatest common divisor of $2a^3x + 4a^2bx + 2ab^2x$ and $4a^2x^3 + 8abx^3 + 4b^2x^3$.

Ans. $2a^2x + 4abx + 2b^2x$.

45. In many cases it is not possible to resolve quantities into their factors by inspection. In order to deduce a rule for finding the Greatest Common Divisor, in any case whatever, it will be necessary to establish the following principle; viz.:

Any quantity, that will divide two other quantities, will also divide the difference between any two multiples of those quantities.

Let P and P' denote any two quantities, both of which are divisible by D; and let m and n be any two factors whatever: then will mP and nP' be any two multiples of P and P', and their difference, $mP - nP'$, will also be divisible by D. Let Q and Q' denote the quotients of P and P' by D: then will

$$mP - nP' = mQD - nQ'D = D(mQ - nQ').$$

The last expression is evidently divisible by D: hence, its equal, $mP - nP'$, must also be divisible by D; which was to be proved.

Hence, we see, that if any polynomial be multiplied by any factor, and if from the result a second polynomial be subtracted any number of times, the remainder will contain all the factors common to the two polynomials; that

is, if the first polynomial be multiplied by any factor, and the result be divided by the second polynomial, the remainder will contain all the factors common to the two polynomials. In the same way, if the second polynomial, or any multiple of it, be divided by the first remainder, the second remainder will contain all the factors common to the polynomials used, and consequently, to the given polynomials; and so on, indefinitely.

Hence, we may write the following rule for finding the greatest common divisor of two quantities:

RULE.

I. *Suppress all the monomial factors of each polynomial; if any factor suppressed is common to the two, set it aside as a factor of the common divisor.*

II. *Multiply the first polynomial by such a factor as will make its first term divisible by the first term of the second polynomial; then divide this result by the second polynomial, continuing the division as far as possible.*

III. *Take the second polynomial as a dividend, and the first remainder as a divisor, and proceed as before; and so on, till a remainder is found that will divide the preceding divisor. This remainder, multiplied by the factors set aside, will give the Greatest Common Divisor.*

EXAMPLES.

1. Find the greatest common divisor of $2x^2 + 5xy + 3y^2$ and $3x^2 + xy - 2y^2$.

Multiplying the first by 3, and dividing by the second, we have the following

OPERATION.

$$\begin{array}{l|l} 6x^2 + 15xy + 9y^2 & 3x^2 + xy - 2y^2 \\ \underline{6x^2 + 2xy - 4y^2} & \underline{2} \\ 13xy + 13y^2 \quad \ldots \text{ 1st } \textit{remainder.} & \end{array}$$

Suppressing the factor $13y$, in the first remainder, and then dividing the second polynomial by $x + y$, we have the following

OPERATION.

$$\begin{array}{l|l} 3x^2 + xy - 2y^2 & x + y \\ \underline{3x^2 + 3xy} & \underline{3x - 2y} \\ -2xy - 2y^2 & \\ \underline{-2xy - 2y^2} & \\ 0 & \end{array}$$

Here, the division is exact; hence, by the rule, $x + y$ is the common divisor sought.

2. Find the greatest common divisor of $x^3 - 5x^2 + 7x - 3$ and $x^2 + x - 12$.

FIRST OPERATION.

$$\begin{array}{l|l} x^3 - 5x^2 + 7x - 3 & x^2 + x - 12 \\ \underline{x^3 + x^2 - 12x} & \underline{x - 6} \\ -6x^2 + 19x - 3 & \\ \underline{-6x^2 - 6x + 72} & \\ 25x - 75 \quad \ldots \text{ 1st } \textit{remainder.} & \end{array}$$

Suppressing the factor 25 in the first remainder, we have the following

OPERATION.

$$\begin{array}{l|l} x^2 + x - 12 & x - 3 \\ \underline{x^2 - 3x} & \underline{x + 4} \\ \quad 4x - 12 & \\ \quad \underline{4x - 12} & \\ \quad 0 & \end{array}$$

Hence, $x - 3$ is the common divisor required.

3. Find the greatest common divisor of $3a^2x^3 - 3a^2y^3$ and $6ax^2 - 6ay^2$.

Suppressing the factor $3a^2$ in the first, and $6a$ in the second, we find a common factor, $3a$, in both, which we set aside as a factor of the divisor sought. Proceeding according to the rule, we have the

FIRST OPERATION.

$$\begin{array}{l|l} x^3 - y^3 & x^2 - y^2 \\ \underline{x^3 - xy^2} & \underline{x} \\ \quad xy^2 - y^3 \text{ . . . 1st } \textit{remainder.} & \end{array}$$

Suppressing the factor y^2, in the first remainder, we have the

SECOND OPERATION.

$$\begin{array}{l|l} x^2 - y^2 & x - y \\ \underline{x^2 - xy} & \underline{x + y} \\ \quad xy - y^2 & \\ \quad \underline{xy - y^2} & \\ \quad 0 & \end{array}$$

Hence, the common divisor sought is $3a(x - y)$.

Find the greatest common divisors of the following:

4. Of $a^2 - 4$ and $a^2 + 4a + 4$. *Ans.* $a + 2$.

5. Of $a^3 - ab^2$ and $a^2 + 2ab + b^2$. *Ans.* $a + b$.

6. Of $x^5 - x^3b^2$ and $x^4 - b^4$. *Ans.* $x^2 - b^2$.

7. Of $x^2 + 2x - 3$ and $x^2 + 5x + 6$. *Ans.* $x + 3$.

8. Of $3x^2y + 3xy^2$ and $3x^2 + 6xy + 3y^2$.
Ans. $3x + 3y$.

9. Of $x^4 + ax^3 - a^3x - a^4$ and $x^4 + a^2x^2 + a^4$.
Ans. $x^2 + ax + a^2$.

10. Of $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.
Ans. $5x^2 - 1$.

46. To find the greatest common divisor of three quantities. First, find that of the first and second; then that of the result and the third quantity.

EXAMPLES.

1. Find the greatest common divisor of $2a^2x^2$, $4x^2y^2$, and $8x^3y$.

The greatest common divisor of $2a^2x^2$ and $4x^2y^2$, is $2x^2$, which exactly divides $8x^3y$; hence, it is the divisor required.

2. Find the greatest common divisor of $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

The greatest common divisor of the first and second, is $x + 4$, and that between this result and the third polynomial, is $x + 4$, which is the divisor sought.

Find the greatest common divisors of the following:

3. Of $3a^nx^{n-1}$, $6a^{2n}x^{n+1}$, and $21a^{n-1}x^{2n}$.
Ans. $3a^{n-1}x^{n-1}$.

4. Of $4ax^2y$, $16abx^2$, and $24acx^2$. *Ans.* $4ax^2$.

5. Of $7a^2 + 7ab$, $4ab + 4b^2$, and $2ac + 2bc$.
Ans. $a + b$.

6. Of $3x^2 - 6x$, $2x^3 - 4x^2$, and $x^2y - 2xy$.
Ans. $x^2 - 2x$.

7. Of $3x^2 + 6xy$, $2xy + 4y^2$, and $4xz + 8zy$.
Ans. $x + 2y$.

8. Of $3a^2 - 3b^2$, $3a^2 + 6ab + 3b^2$, and $3axy + 3bxy$.
Ans. $3(a + b)$.

9. Of $x^2 - 9$, $x^2 - 3x - 18$, and $x^2 + 11x + 24$.
Ans. $x + 3$.

10. Of $x^2 - 3x - 28$ $x^2 - 11x + 28$, and $x^2 - 15x + 56$.
Ans. $x - 7$.

11. Of $x^4 - 5x^2 + 6$, $x^4 - 7x^2 + 12$, and $x^4 + 2x^2 - 15$.
Ans. $x^2 - 3$.

12. Of $x^3 + 5x^2 + 7x + 3$, $x^3 + 3x^2 - x - 3$, and $x^3 + x^2 - 5x + 3$. *Ans.* $x + 3$.

LEAST COMMON MULTIPLE.

47. One quantity is a MULTIPLE of another when it can be divided by it without a remainder. Thus, $8a^2b$, is a multiple of 8, also of a^2, and of b.

48. A quantity is a *Common Multiple* of two or more quantities, when it can be divided by each, separately, without a remainder. Thus, $24a^3x^2$, is a common multiple of $6ax$ and $4a^2x$.

49. The LEAST COMMON MULTIPLE of two or more quantities, is the simplest quantity that can be divided

by each, without a remainder. Thus, $12a^2b^2x^2$, is the least common multiple of $2a^2x$, $4ab^2$, and $6a^2b^2x^2$.

50. When the given quantities can be factored, by any of the methods already given, the Least Common Multiple may be found by the following

RULE.

I. *Resolve each of the quantities into its prime factors.*

II. *Take each factor as often as it enters any one of the quantities, and form the continued product of these factors; it will be the common multiple required.*

EXAMPLES.

1. Find the least common multiple of $6ab^2$ and $18a^2b$.

Factoring, we have,

$$6ab^2 = 2 \times 3abb, \text{ and } 18a^2b = 2 \times 3 \times 3aab.$$

The greatest number of times that the factor 2 enters either quantity, is once, whilst the factor 3 enters twice in the second, b twice in the first, and a twice in the second. Hence, the multiple required is

$$2 \times 3 \times 3aabb = 18a^2b^2.$$

2. Find the least common multiple of $6a^2xy$, $8ax^2$, and $12x^2y^2$.

Factoring, we have,

$$\begin{aligned} 6a^2xy &= 2 \,.\, 3aaxy \\ 8ax^2 &= 2 \,.\, 2 \,.\, 2axx \\ 12x^2y^2 &= 2 \,.\, 2 \,.\, 3xxyy: \end{aligned}$$

hence, the least common multiple is

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot aaxxyy, \text{ or } 24a^2x^2y^2.$$

3. Find the least common multiple of $a^2 - b^2$ and $a^2 - 2ab + b^2$.

Factoring, we have,

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ a^2 - 2ab + b^2 &= (a - b)(a - b); \end{aligned}$$

hence, the least common multiple required is

$$(a - b)(a - b)(a + b), \text{ or } a^3 - ab^2 - a^2b + b^3.$$

Find the least common multiples of the following:

4. Of $15x^2y^2$ and $6x^3y$. *Ans.* $30x^3y^2$.

5. Of $3x^2yz$, $6xy^3$, and $9xyz$. *Ans.* $18x^2y^3z$.

6. Of $3ab^2$, $6ac^3$, $4c^3d$, and b^2c^2. *Ans.* $12ab^2c^3d$.

7. Of $ax - bx$, $ay - by$, and x^2y^2.
Ans. $ax^2y^2 - bx^2y^2$.

8. Of $a - b$, $a^2 - b^2$, and $a^2 - 2ab + b^2$.
Ans. $(a - b)^2 (a + b)$.

9. Of $8x^2(x - y)$, $15x^5(x - y)^2$, and $12x^3(x^2 - y^2)$.
Ans. $120x^5(x^2 - y^2)(x - y)$.

10. Of $2a^2(a + x)$ and $4ax(a - x)$.
Ans. $4a^2x(a^2 - x^2)$.

51. In general, it is not possible to factor quantities by inspection. When the quantities cannot be factored by any of the preceding methods, the Least Common Multiple may still be found.

Let P and P' denote any two quantities, and let D denote their greatest common divisor. Let Q and Q' denote the quotients of P and P' by D. Then shall we have,

$$P = Q \times D, \text{ and } P' = Q' \times D.$$

Since Q and Q' have no common factor, their least common multiple is $Q \times Q'$; consequently the least common multiple of P and P', is $D \times Q \times Q'$; or, since $P = DQ$, we shall have the least common multiple equal to $P \times Q'$.

That is, the least common multiple of two quantities is equal to one of the quantities multiplied by the quotient of the other, by their greatest common divisor. Hence, to find the least common multiple of two quantities, we have the following

RULE.

Find their greatest common divisor; divide one of them by it, and multiply the other by the quotient.

EXAMPLES.

1. Find the least common multiple of $a^3 - x^3$ and $a^2 - x^2$.

Their greatest common divisor is $a - x$. Hence, the least common multiple is,

$$\frac{a^2 - x^2}{a - x} \times (a^3 - x^3), \text{ or } a^4 + a^3x - ax^3 - x^4.$$

2. Find the least common multiple of $2x - 1$ and $4x^2 - 1$.

The greatest common divisor of the two, is $2x - 1$: hence, the multiple required is,

$$\frac{2x - 1}{2x - 1} \times (4x^2 - 1) = 4x^2 - 1.$$

3. Find the least common multiple of $x^2 + 7x + 12$, and $x^2 + 8x + 15$. *Ans.* $(x + 3)(x + 4)(x + 5)$.

When there are more than two quantities, find the least common multiple of the first and second, then that of the result and the third; and so on, to the last.

Find the least common multiples of the following:

4. Of $8a^2$, $12a^3$, and $20a^4$. *Ans.* $120a^4$.

5. Of $x^2 + 5x + 6$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$. *Ans.* $(x^2 + 2x - 8)(x^2 + 5x + 6.)$

6. Of $x - 1$, $x^2 - 1$, and $x^2 + 4x - 5$. *Ans.* $(x^2 - 1)(x + 5)$.

7. Of $10x(x + y)$, $8y(x - y)$, and $5(x^2 - y^2)$. *Ans.* $40xy(x^2 - y^2)$.

8. Of $18x^4(x - y)$, $25x^3(x - y)^2$, and $12x^5(x - y)^3$. *Ans.* $900x^5(x - y)^3$.

9. Of $x^3 - 1$, and $x^2 + x - 2$. *Ans.* $x^4 + 2x^3 - x - 2$.

10. Of $6x^2 - x - 1$, and $2x^2 + 3x - 2$. *Ans.* $6x^3 + 11x^2 - 3x - 2$.

11. Of $a - x$, $a^2 - x^2$, and $a^3 - x^3$. *Ans.* $a^4 + a^3x - ax^3 - x^4$.

12. Of $3x^2 - 11x + 6$, $2x^2 - 7x + 3$, $6x^2 - 7x + 2$. *Ans.* $6x^3 - 25x^2 + 23x - 6$.

CHAPTER IV.

FRACTIONS.

52. If the unit 1 be divided into any number of equal parts, each part is called a FRACTIONAL UNIT Thus, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{7}$, $\frac{1}{b}$, are fractional units.

53. A FRACTION is a fractional unit, or a collection of fractional units. Thus $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{7}$, $\frac{a}{b}$, are fractions.

54. Every fraction is composed of two parts: the *Denominator*, which shows into how many parts the unit 1 is divided; and the *Numerator*, which shows how many of these parts are taken. Thus, in the fraction $\frac{a}{b}$, the denominator b, shows that 1 is divided into b equal parts, and the numerator a, shows that a of these parts are taken. The fractional unit, in all cases, is equal to the reciprocal of the denominator.

55. A DECIMAL FRACTION is one whose denominator is some power of 10. In such fractions, the numerator is expressed by means of the decimal point, which signifies that the denominator is equal to 1, followed by as many 0's as there are decimal places. Thus, the expressions, .034, .0079, are decimal fractions, equivalent to the fractions, $\frac{34}{1000}$, $\frac{79}{10000}$.

56. An ENTIRE QUANTITY, is one which contains no fractional part. Thus, 7, 11, a^3x, $4x^2 - 3y$, are *entire quantities.*

An entire quantity may be regarded as a fraction whose denominator is 1. Thus, $7 = \frac{7}{1}$, $ab = \frac{ab}{1}$.

57. A MIXED QUANTITY, is a quantity containing both entire and fractional parts. Thus, $7\frac{4}{10}$, $8\frac{3}{7}$, $a + \frac{bx}{c}$, are mixed quantities.

58. Let $\frac{a}{b}$ denote any fraction, and q any quantity whatever. From the preceding definitions, $\frac{a}{b}$ denotes that $\frac{1}{b}$ is taken a times; also, $\frac{aq}{b}$ denotes that $\frac{1}{b}$ is taken aq times; that is,

$$\frac{aq}{b} = \frac{a}{b} \times q.$$

Hence, *multiplying the numerator of a fraction by any quantity, is equivalent to multiplying the fraction by that quantity.*

We see, also, that *any quantity may be multiplied by a fraction, by multiplying it by the numerator, and then dividing the result by the denominator.*

59. It is a principle of Division, that the same result will be obtained, if we divide the quantity a by the product of two factors, $p \times q$, as would be obtained by dividing it first by one of the factors, p, and then dividing that result by the other factor, q. That is,

$$\frac{a}{pq} = \left(\frac{a}{p}\right) \div q; \text{ or, } \frac{a}{pq} = \left(\frac{a}{q}\right) \div p.$$

Hence, *multiplying the denominator of a fraction by any quantity, is equivalent to dividing the fraction by that quantity.*

60. Since the operations of Multiplication and Division are the converse of each other, it follows, from the preceding principles,

That, *dividing the numerator of a fraction by any quantity, is equivalent to dividing the fraction by that quantity;*

And, *dividing the denominator of a fraction by any quantity, is equivalent to multiplying the fraction by that quantity.*

61. Since a quantity may be multiplied, and the result divided by the same quantity, without altering the value, it follows that, *both terms of a fraction may be multiplied by any quantity, or both divided by any quantity, without changing the value of the fraction.*

TRANSFORMATION OF FRACTIONS.

62. The *transformation* of a quantity, is the operation of changing its form, without altering its value.

FIRST TRANSFORMATION.

To reduce an entire quantity to a fractional form having a given denominator.

63. Let a be the quantity, and b the given denominator. We have, evidently, $a = \frac{ab}{b}$; hence, the

RULE.

Multiply the quantity by the given denominator, and write the product over this given denominator.

SECOND TRANSFORMATION.

To reduce a fraction to its lowest terms.

64. A fraction is in its *lowest terms*, when the numerator and denominator contain no common factors.

It has been shown, that both terms of a fraction may be divided by the same quantity, without altering its value. Hence, if they have any common factors, we may strike them out.

RULE.

Resolve both terms of the fraction into their prime factors; then strike out all that are common to both.

The same result is attained by dividing both terms of the fraction by their greatest common divisor.

EXAMPLES.

1. Reduce $\frac{14ax}{21ay}$ to its lowest terms.

Factoring both terms, we have,

$$\frac{14ax}{21ay} = \frac{2x \times 7a}{3y \times 7a};$$

Striking out the common factors, 7 and a, or $7a$, we have,

$$\frac{14ax}{21ay} = \frac{2x}{3y}.$$

2. Reduce the fraction $\frac{3a^2 - 3b^2}{4a - 4b}$, to its lowest terms.

We have,

$$\frac{3a^2 - 3b^2}{4a - 4b} = \frac{3(a^2 - b^2)}{4(a - b)} = \frac{3(a + b)(a - b)}{4(a - b)} = \frac{3(a + b)}{4}.$$

Reduce the following fractions to their simplest terms:

3. $\frac{16abx^2}{24a^2b^2x}$. *Ans.* $\frac{2x}{3ab}$.

4. $\frac{12a^2cd}{16abc}$. *Ans.* $\frac{3ad}{4b}$.

5. $\frac{45x^3y^3z}{36abx^2y^2z}$. *Ans.* $\frac{5xy}{4ab}$.

6. $\frac{2(x^2 - y^2)}{x^2 - 2xy + y^2}$. *Ans.* $\frac{2(x + y)}{x - y}$.

7. $\frac{5(a^2 - x^2)}{10(a - x)}$. *Ans.* $\frac{1}{2}(a + x)$.

8. $\frac{x^2 - a^2}{x^2 + 2ax + a^2}$. *Ans.* $\frac{x - a}{x + a}$

9. $\frac{3ax^2 - 3a^2x}{2x^2y - 2axy}$. *Ans.* $\frac{3a}{2y}$.

10. $\frac{3x^2 - 6x}{2xy - 4y}$. *Ans.* $\frac{3x}{2y}$.

11. $\frac{x^2 + 2ax + a^2}{3(x^2 - a^2)}$. *Ans.* $\frac{x + a}{3(x - a)}$.

12. $\frac{x^2 + x - 2}{2x^3 - 3x + 1}$. *Ans.* $\frac{x + 2}{2x^2 + 2x - 1}$.

The last example is solved by finding the greatest common divisor of the two terms, which is $x - 1$, and then dividing both by it. The following examples may be solved in the same manner:

13. $\dfrac{x^2-9}{x^2-x-12}$. *Ans.* $\dfrac{x-3}{x-4}$.

14. $\dfrac{x^2-2ax+a^2}{7x^2y-10axy+3a^2y}$. *Ans.* $\dfrac{x-a}{(7x-3a)y}$.

15. $\dfrac{12x^2-15xy+3y^2}{6x^3-6x^2y+2xy^2-2y^3}$. *Ans.* $\dfrac{12x-3y}{6x^2+2y^2}$.

THIRD TRANSFORMATION.

To reduce a fraction to a mixed quantity.

65. When any term of the numerator is divisible by any term of the denominator, the transformation can be effected by the principles of Division.

RULE.

Perform the indicated division, continuing the operation as far as possible; then write the remainder over the denominator, and annex the result to the quotient found.

EXAMPLES.

1. $\dfrac{a^2+x^2}{a+x} = a-x+\dfrac{2x^2}{a+x}$.

2. $\dfrac{x^2+x-4}{x+2} = x-1-\dfrac{2}{x+2}$.

3. $\dfrac{a^3+x^3}{a^2+2ax+x^2} = a-2x+\dfrac{3x^2}{a+x}$.

4. $\dfrac{x^2+a^2+3-2ax}{x-a} = x-a+\dfrac{3}{x-a}$.

5. $\dfrac{30-11x-44x^2+32x^3}{15+17x-4x^2} = 2-3x+\dfrac{5x^2}{5-x}$.

6. $\dfrac{x^2+3x-25}{x-4} = x+7+\dfrac{3}{x-4}$.

7. $\dfrac{2y^4+19y^2+35}{y^3-3y^2+7y-21} = 2y+6+\dfrac{23}{y-3}$.

8. $\dfrac{4x-x^2+3-y}{4-x} = x+\dfrac{3-y}{4-x}$.

FOURTH TRANSFORMATION.

To reduce a mixed quantity to a fractional form.

66. This transformation is the converse of the preceding, and may always be effected by the following

RULE.

Multiply the entire part by the denominator of the fraction, and add to the product the numerator; write the result over the denominator of the fraction.

EXAMPLES.

1. $a+\dfrac{b}{c} = \dfrac{ac+b}{c}$.

2. $a+\dfrac{ax}{a-x} = \dfrac{a(a-x)+ax}{a-x} = \dfrac{a^2}{a-x}$.

3. $1+\dfrac{c}{x-y} = \dfrac{x-y+c}{x-y}$.

4. $1+\dfrac{a^2+b^2-c^2}{2ab} = \dfrac{(a+b)^2-c^2}{2ab}$.

5. $a+\dfrac{ac+d}{c+d} = \dfrac{2ac+ad+d}{c+d}$.

6. $x+y+\dfrac{x^2+y^2}{x-y} = \dfrac{2x^2}{x-y}$.

7. $ab + cd + \dfrac{abc - c^2d - 2cd^2}{c + 2d} = \dfrac{2ab(c + d)}{c + 2d}$.

8. $xy - ab - \dfrac{2xy^2 - 2aby}{x + y} = \dfrac{(xy - ab)(x - y)}{x + y}$.

FIFTH TRANSFORMATION.

To reduce fractions having different denominators, to equivalent ones having a common denominator.

67. Take the fractions, $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$.

Since both terms of a fraction may be multiplied by the same quantity, without affecting its value, we may multiply both terms of each fraction by the product of the denominators of the others, giving,

$$\frac{a}{b} = \frac{adn}{bdn}, \quad \frac{c}{d} = \frac{cbn}{bdn}, \text{ and } \quad \frac{m}{n} = \frac{mbd}{bdn}.$$

The resulting fractions are equivalent to the given fractions, and have a common denominator, *bdn*. In the same way, other fractions may be treated; hence, the following

RULE.

Multiply each numerator into all the denominators except its own for new numerators, and all the denominators together for a new denominator.

Fractions may always be reduced to a common denominator, by the preceding rule; but, if the denominators have any common factors, it will not be the *least* common denominator.

Take the fractions,

$$\frac{3x}{4a^2}, \quad \frac{2y}{3a}, \text{ and } \frac{z}{6a^3}.$$

The least common multiple of all the denominators is $12a^3$; multiply both terms of the first fraction, by $\frac{12a^3}{4a^2}$, or $3a$; both terms of the second, by $\frac{12a^3}{3a}$, or $4a^2$; and both terms of the third, by $\frac{12a^3}{6a^3}$, or 2; and there will result the fractions,

$$\frac{9ax}{12a^3}, \quad \frac{8a^2y}{12a^3}, \text{ and } \frac{2z}{12a^3},$$

which are equivalent to the given fractions, and have a common denominator. In the same way, other fractions may be transformed. Hence, the following

RULE.

Find the least common multiple of all the denominators, for a common denominator. Divide this least common multiple by each denominator, separately, and multiply the quotients by the corresponding numerators for new numerators.

If there are any entire quantities, they may be regarded as fractions whose denominators are equal to 1.

EXAMPLES.

1. Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{d}{x}$, to a common denominator.

Multiplying both terms of the first by dx, both terms of the second by bx, and both terms of the third by bd, we have,

$$\frac{adx}{bdx}, \quad \frac{bcx}{bdx}, \text{ and } \frac{bd^2}{bdx}.$$

2. Reduce $\frac{a}{x^2-a^2}$, $\frac{b}{x-a}$, and $\frac{c}{x+a}$, to a common denominator.

Here, the least common multiple of the denominators is x^2-a^2. Multiplying both terms of each fraction by the quotient of x^2-a^2, divided by the denominator of that fraction, we have the equivalent fractions,

$$\frac{a}{x^2-a^2}, \quad \frac{b(x+a)}{x^2-a^2}, \quad \text{and} \quad \frac{c(x-a)}{x^2-a^2}.$$

Reduce each of the following fractions to a common denominator:

3. $\frac{x+a}{b}$, $\frac{a}{b}$, and $\frac{a-x}{a}$.

Ans. $\frac{a(x+a)}{ab}$, $\frac{a^2}{ab}$, and $\frac{b(a-x)}{ab}$.

4. $\frac{m}{4a(a+x)}$ and $\frac{n}{4(a^2-x^2)}$.

Ans. $\frac{m(a-x)}{4a(a^2-x^2)}$ and $\frac{na}{4a(a^2-x^2)}$.

5. $\frac{3x}{4}$, $\frac{4}{6}$, and $\frac{12x^2}{15}$. *Ans.* $\frac{45x}{60}$, $\frac{40}{60}$, and $\frac{48x^2}{60}$.

6. $\frac{2b}{15}$, $\frac{3c}{5}$, and $\frac{4d}{25}$. *Ans.* $\frac{10b}{75}$, $\frac{45c}{75}$, and $\frac{12d}{75}$.

7. a, $\frac{3b^2}{4}$, and $\frac{5c^3}{6}$. *Ans.* $\frac{12a}{12}$, $\frac{9b^2}{12}$, and $\frac{10c^3}{12}$.

8. $\frac{x}{1-x}$, $\frac{x^2}{(1-x)^2}$, and $\frac{x^3}{(1-x)^3}$.

Ans. $\frac{x(1-x)^2}{(1-x)^3}$, $\frac{x^2(1-x)}{(1-x)^3}$, and $\frac{x^3}{(1-x)^3}$.

9. $3bx$, $\frac{a}{a+x}$, $\frac{b}{a^2-x^2}$, and $\frac{c}{x}$.

Ans. $\frac{3bx^2(a^2-x^2)}{x(a^2-x^2)}$, $\frac{ax(a-x)}{x(a^2-x^2)}$, $\frac{bx}{x(a^2-x^2)}$, $\frac{c(a^2-x^2)}{x(a^2-x^2)}$.

10. $\frac{cx}{a-x}$, $\frac{dx^2}{a+x}$, and $\frac{x^3}{a+x}$.

Ans. $\frac{cx(a+x)}{a^2-x^2}$, $\frac{dx^2(a-x)}{a^2-x^2}$, and $\frac{x^3(a-x)}{a^2-x^2}$.

11. $\frac{4}{c-x}$, $\frac{5}{x^2}$, and $\frac{6}{x^3}$.

Ans. $\frac{4x^3}{x^3(c-x)}$, $\frac{5x(c-x)}{x^3(c-x)}$, and $\frac{6(c-x)}{x^3(c-x)}$.

12. 4, $\frac{5}{a^2-x^2}$, $\frac{6}{a^2+x^2}$, and $\frac{7}{y}$.

Ans. $\frac{4y(a^4-x^4)}{y(a^4-x^4)}$, $\frac{5y(a^2+x^2)}{y(a^4-x^4)}$, $\frac{6y(a^2-x^2)}{y(a^4-x^4)}$, and $\frac{7(a^4-x^4)}{y(a^4-x^4)}$.

In what follows, we shall suppose all entire and all mixed quantities to be reduced to a fractional form, and all will be treated together as fractions.

ADDITION OF FRACTIONS.

68. Fractions can only be added when they have a common unit, that is, when they have a common denominator. In that case, the numerator of each fraction will indicate the number of times the common fractional unit is taken, in that fraction; and the sum of the numerators will indicate how many times that unit is taken in the entire collection. Hence, the

RULE.

Reduce the fractions to a common denominator; then add the numerators together for a new numerator, and write the sum over the common denominator.

EXAMPLES.

1. Find the sum of a, $\frac{1}{a}$, $\frac{1}{2b}$, and $\frac{3x}{4a^2}$.

Reducing to a common denominator,

$$\frac{4a^3b}{4a^2b}, \quad \frac{4ab}{4a^2b}, \quad \frac{2a^2}{4a^2b}, \quad \frac{3bx}{4a^2b};$$

hence, the sum is $\frac{(4a^3 + 4a + 3x)b + 2a^2}{4a^2b}$.

Find the sums of the following quantities:

2. $\frac{2}{a^2b^3}$, $\frac{3}{a^3b^2}$, and $\frac{4}{a^3b^3}$. *Ans.* $\frac{2a + 3b + 4}{a^3b^3}$.

3. $\frac{2a}{3x^2}$, $\frac{a + 2x}{4x}$, and $\frac{a}{6x}$. *Ans.* $\frac{6x^2 + 5ax + 8a}{12x^2}$.

4. $\frac{x + y}{2}$, and $\frac{x - y}{2}$. *Ans.* x.

5. $\frac{2x}{3}$, $\frac{3x}{5}$, and $\frac{5x}{7}$. *Ans.* $x + \frac{103}{105}x$.

6. $\frac{2x}{1 - x^2}$, and $\frac{1}{x + 1}$. *Ans.* $\frac{1}{1 - x}$.

7. $\frac{2}{(x - 1)^3}$, $\frac{3}{(x - 1)^2}$, and $\frac{4}{x - 1}$.

Ans. $\frac{4x^2 - 5x + 3}{(x - 1)^3}$

8. $\frac{a}{(1+a)(a+x)}$, and $\frac{x}{(1-x)(a+x)}$.

Ans. $\frac{1}{(1+a)(1-x)}$.

9. $\frac{1}{4(1+a)}$, $\frac{1}{4(1-a)}$, and $\frac{1}{2(1-a^2)}$.

Ans. $\frac{1}{1-a^2}$.

10. $\frac{3x-4y}{7}$, $\frac{-2x+y+1}{3}$, and $\frac{15x-4}{12}$.

Ans. $\frac{85x-20y}{84}$

SUBTRACTION OF FRACTIONS.

69. Fractions can only be subtracted when they have the same unit. In that case, the numerator of the minuend, *minus* that of the subtrahend, will indicate the number of times that the common unit is to be taken, in the difference. Hence, the

RULE.

Reduce the fractions to a common denominator; then subtract the numerator of the subtrahend from that of the minuend for a new numerator, and write the remainder over the common denominator.

EXAMPLES.

1. From $\frac{a+2x}{a-2x}$, subtract $\frac{a-2x}{a+2x}$.

By the rule,

$$\frac{a+2x}{a-2x} - \frac{a-2x}{a+2x} = \frac{(a+2x)^2}{a^2-4x^2} - \frac{(a-2x)^2}{a^2-4x^2} = \frac{8ax}{a^2-4x^2}.$$

2. From $4a + \frac{2a}{c}$, subtract $2a - \frac{a - 3b}{c}$.

Ans. $2a + \frac{3(a - b)}{c}$.

3. From $\frac{5x + 3y}{4}$, subtract $\frac{x - 2y}{5}$.

Ans. $\frac{21x + 23y}{20}$.

4. From $\frac{a}{a - x}$, subtract $\frac{x}{a + x}$. *Ans.* $\frac{a^2 + x^2}{a^2 - x^2}$.

5. From $\frac{x + y}{x - y}$, subtract $\frac{x - y}{x + y}$. *Ans.* $\frac{4xy}{x^2 - y^2}$.

6. From $a + \frac{a - x}{a(a + x)}$, subtract $\frac{a + x}{a(a - x)}$.

Ans. $a - \frac{4x}{a^2 - x^2}$.

7. From $3x + \frac{11x - 10}{15}$, subtract $2x + \frac{3x - 5}{7}$.

Ans. $\frac{137x + 5}{105}$.

8. From $\frac{1}{y - z}$, subtract $\frac{1}{y^2 - z^2}$. *Ans.* $\frac{y + z - 1}{y^2 - z^2}$.

MULTIPLICATION OF FRACTIONS.

70. Let $\frac{a}{b}$ and $\frac{c}{d}$, represent any two fractions. It has been shown (Art. 58), that any quantity may be multiplied by a fraction, by first multiplying by the numerator, and then dividing the result by the denominator. To multiply $\frac{a}{b}$ by $\frac{c}{d}$, we first multiply by c,

giving $\frac{ac}{b}$; then, we divide this result by d, which is done by multiplying the denominator by d; this gives for the product, $\frac{ac}{bd}$; that is,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Hence, we have the following

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

EXAMPLES.

1. Multiply $\frac{7x}{5y}$, by $\frac{3a}{4c}$.

The product of the numerators is $21ax$, and of the denominators, $20cy$; hence, by the rule,

$$\frac{7x}{5y} \times \frac{3a}{4c} = \frac{21ax}{20cy}.$$

2. Multiply $\frac{2x}{x-y}$, by $\frac{x^2-y^2}{3}$.

We have, by the rule,

$$\frac{2x}{x-y} \times \frac{x^2-y^2}{3} = \frac{2x(x^2-y^2)}{3(x-y)} = \frac{2x(x+y)(x-y)}{3(x-y)} = \frac{2x(x+y)}{3}.$$

In the above example, after indicating the multiplication, we factored both numerator and denominator, and then struck out the common factor before performing the operation. It will, in general, be found expedient to follow this method.

3. Multiply $2\left(\frac{x+y}{x-y}\right)$, by $\frac{x^2-y^2}{x^2+2xy+y^2}$.

Indicating the operation, factoring, &c., we have,

$$\frac{2(x+y)}{x-y} \times \frac{x^2-y^2}{x^2+2xy+y^2} = \frac{2(x+y)\,(x-y)\,(x+y)}{(x-y)\,(x+y)\,(x+y)} = 2.$$

Find the products of the following quantities:

4. $\frac{3x^2y}{4a}$, and $\frac{2a^2b}{c}$. *Ans.* $\frac{3abx^2y}{2c}$.

5. $\frac{7abf}{3cd}$, and $\frac{4x^2y^2}{3ab^2}$. *Ans.* $\frac{28fx^2y^2}{9bcd}$.

6. $\frac{7x+6}{3}$, and $\frac{2x}{5}$. *Ans.* $\frac{14x^2+12x}{15}$.

7. $\frac{2}{x-y}$, and $\frac{x^2-y^2}{a}$. *Ans.* $\frac{2(x+y)}{a}$.

8. $\frac{ab}{4-x}$, and $\frac{3x^2}{a^2}$. *Ans.* $\frac{3bx^2}{a(4-x)}$.

9. $\frac{x^2-4}{3}$, and $\frac{4x}{x+2}$. *Ans.* $\frac{4x(x-2)}{3}$.

10. $a+\frac{b}{x}$, and $b+\frac{a}{x}$. *Ans.* $\frac{(ax+b)\,(bx+a)}{x^2}$.

11. $3+\frac{x}{4}$, and $x+\frac{4}{x}$. *Ans.* $\frac{(12+x)\,(x^2+4)}{4x}$.

12. $\frac{(a+b)^2}{2x}$, and $\frac{4x^2}{a+b}$. *Ans.* $2x(a+b)$.

13. $\frac{3x^2+1}{2}$, and $\frac{2y}{3}$. *Ans.* $\frac{2y(3x^2+1)}{6}$.

14. $\frac{(x-1)^2}{y^3}$, and $\frac{(x+1)y^2}{x-1}$. *Ans.* $\frac{x^2-1}{y}$.

15. $m + \frac{1}{m} - 1$, and $m + \frac{1}{m} + 1$.

Ans. $m^2 + 1 + \frac{1}{m^2}$.

16. $x - \frac{y^2}{x}$, and $\frac{x}{y} + \frac{y}{x}$. *Ans.* $\frac{x^4 - y^4}{x^2y}$.

17. $\frac{x(a-x)}{a^2 + 2ax + x^2}$, and $\frac{a(a+x)}{a^2 - 2ax + x^2}$.

Ans. $\frac{ax}{a^2 - x^2}$.

18. $\frac{a^2 - x^2}{a+b}$, $\frac{a^2 - b^2}{x(a+x)}$, and $a + \frac{ax}{a-x}$.

Ans. $\frac{a^2(a-b)}{x}$.

19. $x + \frac{2xy}{x-y}$, and $x - \frac{2xy}{x+y}$. *Ans.* x^2.

20. $\frac{a^3 - x^3}{a^3 + x^3}$, $\frac{a^2 - x^2}{a^2 + x^2}$, $\frac{a-x}{a+x}$, and $\frac{a^2 - ax + x^2}{a^2 + ax + x^2}$.

Ans. $\frac{(a-x)^3}{a^3 + a^2x + ax^2 + x^3}$.

21. $x^2 + x + 1$, by $\frac{1}{x^2} - \frac{1}{x} + 1$. *Ans.* $x^2 + 1 + \frac{1}{x^2}$.

22. $x + 1 + \frac{1}{x}$, by $x - 1 + \frac{1}{x}$. *Ans.* $x^2 + 1 + \frac{1}{x^2}$.

23. $\frac{2a - b}{4a}$, by $\frac{6a - 2b}{b^2 - 2ab}$. *Ans.* $\frac{b - 3a}{2ab}$.

24. $\frac{a}{a+b} + \frac{b}{a-b}$, by $\frac{a}{a+c} - \frac{b}{b+c}$.

Ans. $\frac{(a^2 + b^2)c}{(a+b)(a+c)(b+c)}$.

DIVISION OF FRACTIONS.

71. Since $\frac{p}{q} = p \times \frac{1}{q}$, it follows that, dividing by a quantity is equivalent to multiplying by its reciprocal. But the reciprocal of a fraction, $\frac{c}{d}$, is $\frac{d}{c}$ (Art. 14); consequently, to divide any quantity by a fraction, we invert the terms of the divisor, and multiply by the resulting fraction. Hence,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc};$$

Whence, the following rule, for dividing one fraction by another:

RULE.

Invert the terms of the divisor, and multiply the dividend by the resulting fraction.

The same remarks as were made on *factoring* and reducing, under the head of Multiplication, are applicable in Division.

EXAMPLES.

1. Divide $\frac{3a^2}{a^2 - b^2}$, by $\frac{a}{a + b}$.

By the rule, we have,

$$\frac{3a^2}{a^2-b^2} \div \frac{a}{a+b} = \frac{3a^2 \times (a+b)}{(a^2-b^2)a} = \frac{3a \,.\, a(a+b)}{a(a+b)\,(a-b)} = \frac{3a}{a-b}.$$

Divide the following fractions:

2. $\frac{3x}{2x - 2}$, by $\frac{2x}{x - 1}$. *Ans.* $\frac{3}{2}$.

3. $\frac{(x+y)^2}{x-y}$, by $\frac{x+y}{(x-y)^2}$. *Ans.* x^2-y^2.

4. $x+\frac{x}{x-1}$, by $x-\frac{x}{x-1}$. *Ans.* $\frac{x}{x-2}$.

5. $\frac{x^3-3x^2a+3xa^2-a^3}{x+a}$, by $\frac{x-a}{x+a}$. *Ans.* $(x-a)^2$

6. $\frac{x^4-y^4}{a^3+b^3}$, by $\frac{x-y}{a^2-ab+b^2}$. *Ans.* $\frac{x^3+x^2y+xy^2+y^3}{a+b}$.

7. $x^2+2+\frac{1}{x^2}$, by $\frac{x}{a}+\frac{1}{ax}$. *Ans.* $\frac{ax^2+a}{x}$

8. $\frac{2ax+x^2}{c^3-x^3}$, by $\frac{x}{c-x}$. *Ans.* $\frac{2a+x}{c^2+cx+x^2}$.

9. $x+y+\frac{x^2}{y}$, by $x+y+\frac{y^2}{x}$. *Ans.* $\frac{x}{y}$.

10. $\frac{x^2-9}{x^2+4x+4}$, by $\frac{x+3}{x+2}$. *Ans.* $\frac{x-3}{x+2}$.

11. 1, by $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. *Ans.* $\frac{abc}{ab+ac+bc}$.

12. $a^4-\frac{1}{a^4}$, by $a-\frac{1}{a}$. *Ans.* $a^3+a+\frac{1}{a}+\frac{1}{a^3}$.

MISCELLANEOUS EXAMPLES.

Simplify the following expressions:

1. $\frac{a^2+ab+b^2}{2}+\frac{a^2-ab+b^2}{2}$. *Ans.* a^2+b^2.

2. $\frac{1}{a-1}-\frac{2a}{a^2+1}+\frac{1}{a+1}$. *Ans.* $\frac{4a}{a^4-1}$.

3. $\frac{y-1}{2}+\frac{y-2}{3}+\frac{y+7}{6}$. *Ans.* y.

4. $\dfrac{2x}{x^2-y^2}+\dfrac{1}{x+y}-\dfrac{1}{x-y}$. *Ans.* $\dfrac{2}{x+y}$.

5. $\dfrac{x-y}{y}+\dfrac{2x}{x-y}-\dfrac{x^3+x^2y}{x^2y-y^3}$. *Ans.* $\dfrac{y}{x-y}$.

6. $\left\{x+\dfrac{y-x}{1+xy}\right\}\div\left\{1-x\dfrac{y-x}{1+xy}\right\}$. *Ans.* y.

7. $\dfrac{3}{2y-3}-\dfrac{2y+15}{4y^2+9}-\dfrac{2}{2y+3}$. *Ans.* $\dfrac{18(2y+15)}{16y^4-81}$.

8. $\dfrac{x}{1-x}-\dfrac{x^2}{(1-x)^2}+\dfrac{x^3}{(1-x)^3}$. *Ans.* $x+\dfrac{x^4}{(1-x)^3}$.

9. $\left\{\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right\}\div\left\{\dfrac{x+2y}{y}-\dfrac{x}{x+y}\right\}$.

Ans. 1.

10. $\dfrac{x^2-9x+20}{x^2-6x}\times\dfrac{x^2-13x+42}{x^2-5x}$.

Ans. $\dfrac{(x-4)(x-7)}{x^2}$.

11. $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}+\dfrac{1}{x^n+1}-\dfrac{1}{x^n-1}$.

Ans. $x^{2n}+2$.

12. $\dfrac{a^2-ax}{bc+bx}\times\dfrac{4(a+x)}{3(c-x)}$. *Ans.* $\dfrac{4a(a^2-x^2)}{3b(c^2-x^2)}$.

13. $\dfrac{a}{b}-\dfrac{a^2-b^2}{b^2}x+\dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}$. *Ans.* $\dfrac{a+bx}{b+ax}$.

14. $\dfrac{a^4-2a^2x^2+x^4}{a^3x+ax^3}\div\left\{\dfrac{a+x}{a}\times\dfrac{a-x}{x}\right\}$.

Ans. $\dfrac{a^2-x^2}{a^2+x^2}$

15. $\left\{a+\dfrac{2ax-1}{b}\right\}\div\dfrac{x-a}{ax+1}$.

Ans. $\dfrac{a^2(bx+2x^2)+a(x+b)-1}{b(x-a)}$

EXPLANATION OF THE SYMBOLS, 0, ∞, AND $\frac{0}{0}$.

72. The symbol 0, is called *Zero.* It stands for *no quantity*, or for *a quantity less than any assignable quantity.* If we subtract a from a, the remainder is *no quantity*, a result which is indicated thus, $a - a = 0$. If, in the fraction, $\frac{a}{x}$, we suppose a to remain invariable, and finite in value, whilst x goes on increasing, the value of the fraction will go on diminishing; and when x becomes very great, with respect to a, the value of the fraction becomes very small; and finally, when x becomes *greater* than any assignable quantity, or *infinitely great*, the value of the fraction becomes *less* than any assignable quantity, or *infinitely small.* This value of the fraction we call Zero, and denote it by the same symbol, 0. The first *zero*, is absolute, and is the *naught* of Arithmetic, the *arithmetical zero;* the second, is the *algebraic zero.*

73. The symbol ∞, is called *infinity.* It stands for a quantity greater than any assignable quantity. If, in the same fraction, $\frac{a}{x}$, we suppose x to diminish, the value of the fraction will increase; and when x becomes exceedingly small, the value of the fraction will become exceedingly great; and finally, when the value of x becomes *less* than any *assignable quantity*, the value of the fraction becomes *greater* than *any assignable quantity.* This value of the fraction we call *infinity*, and denote it by the symbol ∞. This is the *algebraic infinity*, not the infinity of common language.

Hence, we say, in Algebra, that *the quotient of a finite quantity by zero is infinity*, and *the quotient of a finite*

quantity by infinity is zero; that is, $\frac{a}{0} = \infty$, and $\frac{a}{\infty} = 0$.

74. The symbol $\frac{0}{0}$, is called the symbol of *indetermination.* It signifies that a quantity has an infinite number of values. We shall see, hereafter, that $\frac{0}{0}$, forms the proper answer to many algebraic questions. Where there are fewer conditions given than there are required quantities, there will, in general, be an infinite number of solutions, and this fact will be indicated by the values of the required quantity taking the form $\frac{0}{0}$.

75. The value of a fraction sometimes reduces to the form $\frac{0}{0}$, for a particular supposition, in consequence of the existence of a common factor in both terms, which reduces to 0, for the particular hypothesis. This case must not be confounded with the case of indetermination. Thus, the fractions,

$$\frac{b(a-b)(a-b)}{2a(a+b)(a-b)}, \quad \frac{2a(a-b)}{3b(a-b)}, \text{ and } \frac{2b(a-b)}{a(a-b)(a-b)},$$

all reduce to the form $\frac{0}{0}$, when $a = b$, because the factor $a - b$ becomes 0, in that particular case. If this factor be stricken out before making the supposition, $a = b$, the fractions will become respectively 0, $\frac{2}{3}$, and ∞. Before deciding, therefore, upon the nature of the symbol $\frac{0}{0}$, we must first ascertain whether it results from a common factor, in both terms, which reduces to 0 for the supposition made; if it does not it is the true symbol of indetermination.

The fractions just considered are called *vanishing fractions*. From what has preceded, we have the following rule for finding the true value of a vanishing fraction for that supposition which causes it to vanish:

RULE.

Suppress all the factors common to the two terms, and then make the supposition.

EXAMPLES.

1. Find the value of $\frac{x^4 - y^4}{x^2 - y^2}$, when $x = y$.

By the rule,

$$\frac{x^4 - y^4}{x^2 - y^2} = \frac{(x^2 - y^2)(x^2 + y^2)}{x^2 - y^2} = x^2 + y^2;$$

which, when $x = y$, reduces to $2y^2$: this is expressed algebraically as follows:

$$\left\{\frac{x^4 - y^4}{x^2 - y^2}\right\}_{x = y} = 2y^2.$$

2. Find the value of $\left\{\frac{7x(x^4 - 1)}{(1 + x)(x - 1)}\right\} x = 1.$

Ans. 14.

3. Find the value of $\left\{\frac{3(a - b)^2}{4(a^2 - b^2)}\right\} a = b.$

Ans. 0.

4. Find the value of $\left\{\frac{2(a^4 - x^4)}{3(a - x)^2}\right\} a = x.$

Ans. ∞.

CHAPTER V.

EQUATIONS OF THE FIRST DEGREE.

76. AN EQUATION is the expression of equality between two quantities. Thus,

$$x = b + c,$$

is an equation, expressing the fact that the quantity x, is equal to the sum of the quantities b and c.

77. Every equation is composed of two parts, connected by the sign of equality. These parts are called *members:* the part on the left of the sign of equality, is called the *first member;* that on the right, the *second member.* Thus, in the equation,

$$x + y = a - c,$$

$x + y$, is the first member, and $a - c$, the second member.

78. Equations are divided into two classes: those containing *but one* unknown quantity, and those containing *more than one* unknown quantity. Each of these classes is subdivided into degrees. In the first class, the degree is determined by the exponent of the highest power of the unknown quantity, in any term; in the second class, the degree is determined by the highest sum of the exponents of the unknown quantities, in any term.

Thus,

$$bx = c, \qquad bx + cy = d,$$

are equations of the *first degree;*

$$x^2 + 2px = q, \qquad x^2 + axy + y^2 = m,$$

are equations of the *second degree;*

$$ax^3 + bx^2 + cx = d, \qquad x^3 + 2x^2y + 3yx + 4y = 5,$$

are equations of the *third degree;*

$$x^n + px^{n-1} + qx^{n-1} = s, \quad x^{n-2}y^2 + ax^{n-2}y + bxy^{n-1} = d,$$

are equations of the n^{th} *degree.*

We shall first consider equations of the first degree, containing but one unknown quantity.

79. The TRANSFORMATION of an equation, is the operation of changing its form, without destroying the equality of its members.

80. The SOLUTION of an equation, is the operation of finding such a value for the unknown quantity, as will *satisfy* the equation; that is, such a value as, being substituted for the unknown quantity, will render the two members equal. This is called a ROOT of the equation.

81. An AXIOM is a self-evident proposition.

82. The solution of an equation is effected by successive transformations, which transformations depend upon the following axioms:

1. If equal quantities be added to both members of an equation, the equality will not be destroyed.

2. If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.

3. If both members of an equation be multiplied by the same quantity, the equality will not be destroyed.

4. If both members of an equation be divided by the same quantity, the equality will not be destroyed.

5. Like powers of the two members of an equation are equal.

6. Like roots of the two members of an equation are equal.

83. Two principal transformations are employed in the solution of equations of the first degree; *clearing of fractions*, and *transposing*.

CLEARING OF FRACTIONS.

84. Take the equation,

$$\frac{4x}{5} + \frac{6}{10} = \frac{3}{4}.$$

The least common multiple of all the denominators is 20. If we multiply both members of the equation by 20 (Axiom 3), each term can be reduced to an entire form, giving,

$$16x + 12 = 15.$$

In the same manner, any equation may be transformed. Hence, for clearing of fractions, we have the following

RULE.

Find the least common multiple of the denominators, and multiply both members of the equation by it, reducing the fractional to entire terms.

The reduction will be effected, if we divide the least common multiple by each of the denominators, and then

multiply the corresponding numerator, dropping the denominator.

The transformation may be effected by multiplying each numerator into the product of all the denominators except its own, omitting denominators.

EXAMPLES.

1. Clear the equation, $\frac{x}{4} - \frac{2x}{3} = \frac{5}{6}$, of fractions.

The least common multiple of the denominators is 12. Multiplying both members by 12 (Axiom 3), and reducing to entire terms, we have,

$$3x - 8x = 10.$$

2. Clear the equation, $\frac{7x}{5} - \frac{2x}{3} = \frac{5}{2}$, of fractions.

Multiplying each numerator into all the denominators except its own, and, omitting denominators, we have,

$$42x - 20x = 75.$$

Clear the following equations of fractions:

3. $\frac{x}{7} - \frac{3x}{2} + \frac{x}{4} = 5.$ *Ans.* $4x - 42x + 7x = 140.$

4. $\frac{13x}{12} + \frac{4}{3} = 6x.$ *Ans.* $13x + 16 = 72x.$

5. $\frac{x-4}{3} + \frac{x-2}{6} = \frac{5}{3}.$ *Ans.* $2x - 8 + x - 2 = 10.$

6. $-\frac{x-4}{3} - \frac{x-2}{6} = \frac{5}{3}.$ *Ans.* $-2x+8-x+2=10.$

7. $\frac{x-3}{12} - \frac{3x-4}{21} = 8.$

Ans. $7x - 21 - 12x + 16 = 672.$

8. $\frac{x}{3-x} + 4 = \frac{3}{5}$. *Ans.* $5x + 60 - 20x = 9 - 3x$.

TRANSPOSING.

85. TRANSPOSITION is the operation of changing a term from one member to the other, without destroying the equality of the members.

Take the equation,

$$3x + 4 = 5 - 6x.$$

If we add to both members of the equation the quantity $6x - 4$, (Axiom 1), we shall have,

$$3x + 4 + 6x - 4 = 5 - 6x + 6x - 4.$$

Which reduces to

$$3x + 6x = 5 - 4.$$

Comparing this with the given equation, we see that 4 has been transposed to the second member, and $-6x$ to the first member, by changing their signs. In like manner, any term may be transposed. Hence, the following rule for transposing.

RULE.

Any term may be transposed from one member to the other, by changing its sign.

Transpose the unknown terms to the first, and the known terms to the second members, in the following

EXAMPLES.

1. $13x + 16 = 7x + 20$. *Ans.* $13x - 7x = 20 - 16$.
2. $2x - 8 = 10 - x$. *Ans.* $2x + x = 10 + 8$.

3. $7x - 21 = -12x + 1$. *Ans.* $7x + 12x = 1 + 21$.

4. $5x + 60 = 9 - 3x$. *Ans.* $5x + 3x = 9 - 60$.

5. $3x - 7 = -x - 8$. *Ans.* $3x + x = 7 - 8$.

SOLUTION OF EQUATIONS.

86. Take the equation,

$$\frac{2x}{3} + 7 = \frac{5(x-1)}{10} + 8.$$

Clearing of fractions (Art. 83), and performing the operations indicated, we have

$$20x + 210 = 15x - 15 + 240.$$

Transposing all the unknown terms to the first member, and the known terms to the second member (Art. 84), we have,

$$20x - 15x = 240 - 15 - 210.$$

Reducing the terms in the two members, we have

$$5x = 15.$$

Dividing both members by the coefficient of x, we have,

$$x = 5.$$

In the same way, all equations of the first degree, containing but one unknown quantity, may be solved. Hence, the following

RULE.

I. *Clear the equation of fractions, and perform all the indicated operations.*

II. *Transpose all of the unknown terms to the first member, and all the known terms to the second member.*

III. *Reduce all the terms in the first member to a single term, one factor of which is the unknown quantity; the other factor will be the algebraic sum of its coefficients.*

IV. *Divide both members by the coefficient of the unknown quantity: the second member will be the value of the unknown quantity.*

EXAMPLES.

1. Solve the equation,

$$5x - \frac{6x + 3}{11} = \frac{7x + 15}{2} - 3.$$

Clearing of fractions, $110x - 12x - 6 = 77x + 165 - 66$;

transposing and reducing, $21x = 105$;

dividing by 21, $x = 5$.

2. Solve the equation,

$$\frac{b}{ax} - a^2 = b^2 - \frac{a}{bx}.$$

Clearing of fractions, $b^2 - a^3bx = ab^3x - a^2$;

transposing and reducing, $-(a^3b + ab^3)x = -(a^2 + b^2)$;

dividing by $-(a^3b + ab^3)$, $x = \frac{a^2 + b^2}{a^3b + ab^3} = \frac{1}{ab}$.

Solve the following equations:

3. $\frac{3x - 4}{2} = \frac{x}{2} + \frac{x}{4} - \frac{1}{2}$. *Ans.* $x = 2$.

4. $\frac{x}{8} - 1 + \frac{x}{12} - \frac{x + 5}{4} = -\frac{11}{4}$. *Ans.* $x = 12$.

5. $\frac{x+a}{b} - \frac{x}{a} = 1.$ *Ans.* $x = -a.$

6. $\frac{x-1}{2} + \frac{x-2}{3} - \frac{x-3}{4} = 6$ *Ans.* $x = 11.$

7. $\frac{x}{3} - \frac{x}{4} - \frac{1}{2} = \frac{x}{5} - \frac{x}{6}.$ *Ans.* $x = 10.$

8. $\frac{3x-1}{7} + \frac{6-x}{4} - \frac{2x-4}{12} = \frac{54-x}{28}.$

Ans. $x = 5.$

9. $\frac{5x-7}{3} - \frac{3x-2}{7} = \frac{x-5}{4}.$ *Ans.* $x = \frac{67}{83}.$

10. $\frac{x}{8} - \frac{2(x-1)}{5} = \frac{3x-4}{15} + \frac{x}{12}.$ *Ans.* $x = \frac{80}{67}.$

11. $\frac{x-a}{3} - \frac{2x-3b}{5} - \frac{a-x}{2} = 10a + 11b.$

Ans. $x = 25a + 24b.$

12. $\frac{6x+a}{4x+b} = \frac{3x-b}{2x-a}.$ *Ans.* $x = \frac{a^2-b^2}{b-4a}.$

13. $\frac{ax-b}{4} + \frac{a}{3} = \frac{bx}{2} - \frac{bx-a}{3}.$

Ans. $x = \frac{3b}{3a-2b}.$

14. $\frac{a+c}{a+x} + \frac{a-c}{a-x} = \frac{2b^2}{a^2-x^2}.$ *Ans.* $x = \frac{a^2-b^2}{c}.$

15. $\frac{6x+13}{15} - \frac{3x+5}{5} = \frac{2x}{5}.$ *Ans.* $x = -\frac{2}{3}.$

16. $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x-19}{2}.$ *Ans.* $x = 23\frac{1}{4}.$

17. $10\left(x + \frac{1}{2}\right) - 6x\left(\frac{1}{x} - \frac{1}{3}\right) = 23.$ *Ans.* $x = 2.$

18. $\frac{x}{12} - \frac{8-x}{8} - \frac{5+x}{4} + \frac{11}{4} = 0.$

Ans. $x = 12.$

19. $3x - 4 - \frac{4}{5} \times \frac{7x-9}{3} = \frac{4}{5}\left(6 + \frac{x-1}{3}\right).$

Ans. $x = 7\frac{1}{13}.$

20. $\frac{4x}{5-x} - \frac{20-4x}{x} = \frac{15}{x}.$ *Ans.* $x = 3\frac{2}{11}.$

21. $\frac{7x+5}{23} + \frac{9x-1}{10} - \frac{x-9}{5} + \frac{2x-3}{15} = 23\frac{1}{3}.$

Ans. $x = 19.$

22. $\frac{ax}{b} + \frac{cx}{f} + g = qx + \frac{cx}{f} + h.$

Ans. $x = \frac{b(h-g)}{a-bq}.$

23. $\frac{10x+17}{18} - \frac{12x+2}{11x-8} = \frac{5x-4}{9}.$ *Ans.* $x = 4.$

24. $\frac{1}{7}\left(x - \frac{1}{2}\right) - \frac{1}{5}\left(\frac{2}{3} - x\right) = \frac{43}{30}.$ *Ans.* $x = \frac{43}{9}.$

25. $\frac{2x+1}{29} - \frac{402 - 3x}{12} = 9 - \frac{471 - 6x}{2}.$

Ans. $x = 72.$

PROBLEMS.

87. A PROBLEM is a question proposed, requiring a *solution.*

88. The SOLUTION of a problem is the operation of finding a quantity, or quantities, that will satisfy the given conditions.

The solution of a problem consists of two parts; the

statement, and the *solution of the equation or equations of the problem.*

The STATEMENT consists in translating the conditions of the problem into algebraic language, the resulting equations being called *the equations of the problem.* The solution of the equations is made by the general rules for solving equations.

The statement is made by representing the unknown quantities of the problem by some of the final letters of the alphabet, and then operating upon these so as to comply with the conditions of the problem. The method of stating a problem will be best learned from practical examples.

1. What number is that to which if its fifth part be added, the sum will be equal to 24?

Let x denote the number.

Then will $\frac{x}{5}$ denote its fifth part.

From the conditions of the problem,

$$x + \frac{x}{5} = 24;$$

Clearing of fractions, $5x + x = 120,$

reducing, $6x = 120,$

dividing by 6, $x = 20,$ the number required.

2. The sum of two numbers is 30, and their difference 6. What are the numbers?

Let x denote one number.

Then will $30 - x$, denote the other.

From the conditions,

$$(30 - x) - x = 6;$$

transposing and reducing, $-2x = -24;$

dividing by -2, $x = 12$

$\therefore\ 30 - x = 18$ } the two numbers.

3. Two couriers start from points distant 200 miles, and travel towards each other. The first travels 9 miles per hour, and the second $8\frac{1}{2}$ miles per hour. How long before they will meet, and how far will each have travelled?

Let x denote the number of hours.

Then will $9x$ denote the distance the first travels,

and $8\frac{1}{2}x$ denote the distance the second travels.

From the conditions,

$$9x + 8\tfrac{1}{2}x = 200.$$

Solving,

$x = 11\frac{3}{7}$, the number of hours required.

$\therefore\ 9x = 102\frac{6}{7}$, the number of miles the first travels.

$8\frac{1}{2}x = 97\frac{1}{7}$, the number of miles the second travels.

4. A hare starts 50 leaps before a dog, and makes 4 leaps to the dog's 3; but 2 of the dog's leaps are equal to 3 of the hare's. How many leaps must the dog make to overtake the hare?

Let x denote the number of leaps that the dog makes. Then will $\frac{4x}{3}$, denote the number that the hare makes in the same time; and if we take the length of the hare's leap as the unit of distance, the whole distance to be passed over by the dog, will be denoted by $\frac{4x}{3} + 50$. But the dog passes over a distance of $\frac{3}{2}$ units at each leap; hence, in x leaps, he will pass over a distance denoted by $\frac{3}{2}x$. But, from the conditions of the problem, these two quantities are equal; hence,

$$\frac{4x}{3} + 50 = \frac{3x}{2}.$$

$$\therefore\ x = 300. \quad \textit{Ans.}$$

5. A can do a piece of work in 9 days, and B can do the same work in 10 days. In how many days can they both do it together?

Let x denote the number of days required.

If we denote the work by 1, A can do $\frac{1}{9}$ of it in 1 day, and in x days he can do $\frac{x}{9}$ of it. B can do $\frac{1}{10}$ of it in 1 day, and in x days he can do $\frac{x}{10}$ of it. But, from the conditions, the sum of these two will be equal to the entire work; that is,

$$\frac{x}{9} + \frac{x}{10} = 1,$$

$$\therefore x = 4\tfrac{14}{19}. \quad \textit{Ans.}$$

6. If to a certain number its half and its eighth part be added, the sum will be equal to 78. What is the number? *Ans.* 48.

7. A bought a horse, harness, and wagon, for 250 dollars: he gave for the harness one fourth as much as for the horse, and for the wagon as much as for the horse and harness together. What did he give for each?

Ans. \$100 for the horse, \$25 for the harness, and \$125 for the wagon.

8. A drover sold from a flock of sheep one half, and two more; he then sold half that remained, and two more, and then he had 22 left. How many had he at first? *Ans.* 100.

9. If from 3 times a certain number we subtract 8, half the remainder will be equal to the number itself diminished by 2. What is the number? *Ans.* 4.

10. Ten years ago, a boy's age was $\frac{1}{10}$ of his

father's; now, it is $\frac{1}{4}$ of it. What is the father's age now? *Ans.* 60 years.

11. The sixth part of a number added to its eighth part gives 56. What is the number? *Ans.* 192.

12. Two boys had, together, 35 marbles. One fourth of the number that the first had was equal to one third of the number that the second had. How many had each? *Ans.* 20 and 15.

13. A man spent half of his money, and afterwards lost one third of what he had left, when he found that he had remaining $30. How much had he at first? *Ans.* $90.

14. What number is that, from which if 5 be subtracted, one half the remainder is equal to 15? *Ans.* 35.

15. Divide $116 amongst three persons, so that the second shall have two thirds as much as the first, and the third shall have two fifths as much as the second. *Ans.* $60, $40, and $16.

16. A wheat field yielded 72 bushels, which was divided between landlord and tenant in such a way, that for every five bushels that the landlord received, the tenant got seven. How many bushels did the tenant receive? *Ans.* 42.

17. The half of a number exceeds its third part by 8. What is the number? *Ans.* 48.

18. A sum of money is divided between A, B, and C, so that A has $8, B has as much as A, together with one fifth as much as C, and C has as much as A and B together. How much has C? *Ans.* $20.

19. There are 180 sheep in two flocks. If 20 be

taken from the second and added to the first flock, the first flock will then contain twice as many as the second. How many sheep are there in each flock?

Ans. 100 and 80.

20. A post $\frac{1}{3}$ in the mud, $\frac{1}{7}$ in the water, and 10 feet in the air. What is its entire length?

Ans. 28 feet.

21. There are two numbers whose difference is 8, and the first is 5 times the second. What are the numbers?

Ans. 10 and 2.

22. A merchant gains 14 per cent. on his capital, when he finds that he has $8436. What was his capital?

Ans. $7400.

23. A has 3 times as much money as B; but if A were to give to B $100, B would then have 3 times as much as A. How much have they each?

Ans. A $150 and B $50.

24. A laborer was engaged for 30 days, on condition that for every day he labored, he was to receive $2, and for every day he was idle, he was to forfeit $1. At the end of the time he received $21. How many days did he labor? *Ans.* 17.

25. A is twice as old as B, but 10 years ago he was three times as old. How old is B now.

Ans. 20 years.

26. Find that number which, being increased by 9, the result divided by 2, the quotient diminished by 7, the result will be 20. *Ans.* 45.

27. Divide the number 37 into three parts, such that the first shall be 3 less than the second, and the second 5 greater than the third. *Ans.* 12, 15, and 10.

28. A man spends $\frac{2}{3}$ of his income for board, $\frac{2}{3}$ of the remainder for clothing, and has remaining \$70. What is his income? *Ans.* \$630.

29. Divide 1000 into two parts such, that one of them shall be $\frac{3}{5}$ of the other. *Ans.* 375 and 625.

30. A person after spending 50 dollars more than half of his income, had remaining 125 dollars more than a third of it. How much was his income?

Ans. 1050 dollars.

31. In a naval action $\frac{1}{3}$ of a fleet was taken, $\frac{1}{6}$ of it sunk, and 2 ships burnt; $\frac{1}{7}$ of the remainder were afterwards lost in a storm, when 24 ships were left. How many ships were there in the fleet? *Ans.* 60.

32. A sum of 990 dollars was divided between A, B, and C. B received $\frac{4}{5}$ as much as A, and C $\frac{5}{6}$ as much as A and B together. How many dollars did each receive? *Ans.* A, 300; B, 240; and C, 450.

33. A courier A starts 1165 of his own steps ahead of a courier B, and takes 5 steps whilst B takes but 4; now if 3 steps of B are equal to 4 of the courier A, how many steps must B make to overtake A?

Ans. 13980.

34. The hands of a clock are together at 12 o'clock, when are they next together? *Ans.* At 1 h. $5\frac{5}{11}$ m.

35. A grazier spent $\frac{1}{12}$ of his money for horses, $\frac{1}{3}$ for oxen, and $\frac{3}{10}$ of the remainder for sheep, when he had 980 dollars left. How many dollars had he originally?

Ans. 2400.

36. Divide the number 240 into two parts, such that 7 times the first shall equal 5 times the second.

Ans. 100 and 140.

37. In a garrison of 2400 men, there are 3 times as many cavalry as artillery, and twice as many infantry as artillery and cavalry together. How many are there of each kind?

Ans. 200 artillery, 600 cavalry, and 1600 infantry.

38. Divide 21000 dollars between A, B, C, and D, so that A's part shall be $\frac{2}{3}$ of B's, B's part $\frac{4}{5}$ of C's, and C's part $\frac{6}{7}$ of D's. How many dollars will each receive?

Ans. A, 3200; B, 4800; C, 6000; D, 7000.

39. A capital was put out at $6\frac{1}{2}$ per cent. for one year, when the capital and interest together amounted to 1917 dollars. How many dollars were there in the capital?

Ans. 1800.

40. A boatman rows with the tide 42 miles in 3 hours. In returning, the tide is but $\frac{2}{5}$ as strong, and it takes $10\frac{1}{2}$ hours to row the same distance. At what rate per hour did the tide run in each case.

Ans. 6 and 4 miles.

41. A cistern can be filled by two cocks; the first would fill it in 70 minutes, and the second in 80 minutes. In how many minutes would they both fill it together?

Ans. $37\frac{1}{3}$.

EQUATIONS OF THE FIRST DEGREE, CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

86. If we have a single equation, containing two unknown quantities, as

$$2x + 3y = 14,$$

we may find the value of one of them in terms of the other, as follows:

$$x = \frac{14 - 3y}{2} \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Now, if the value of y is unknown, that of x will also be unknown. Hence, from this equation alone, the value of x cannot be determined. If now, we have a second equation,

$$3x + 2y = 11,$$

we may, in like manner, find the value of x in terms of y,

$$x = \frac{11 - 2y}{3} \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

If now, the values of x and y are the same in the two equations, we shall have the second members of (1) and (2) equal to each other; giving the equation,

$$\frac{14 - 3y}{2} = \frac{11 - 2y}{3}, \quad \text{or} \quad 42 - 9y = 22 - 4y.$$

From which we find $y = 4$; and substituting this value for y, in either of the equations (1), or (2), we find $x = 1$.

Such equations are called *simultaneous*. Hence,

SIMULTANEOUS EQUATIONS are those in which the values of the unknown quantities are the same in both.

We have seen that it requires two simultaneous equations, containing two unknown quantities, to determine the values of the unknown quantities. In the same way, it could be shown that it would require three equations containing three unknown quantities, four equations containing four unknown quantities, and so on, to determine the values of the unknown quantities. In general, there must be as many equations as there are unknown quantities. The equations necessary to determine any number of unknown quantities, constitute a *group of simultaneous equations.*

Such equations are solved by *successive eliminations.*

ELIMINATION.

90. ELIMINATION is the operation of combining two equations in such a manner as to get rid of one of the unknown quantities which enter them.

There are three principle methods of elimination, by *Addition or Subtraction; by Substitution;* and *by Comparison.*

1. BY ADDITION OR SUBTRACTION.

91. Take the equations,

$$7x + 6y = 20 \quad . \quad . \quad . \quad . \quad . \quad (1.)$$
$$9x - 4y = 14 \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Multiplying both members of (1) by 4, and of (2) by 6 (Axiom 3),

$$28x + 24y = 80 \quad . \quad . \quad . \quad . \quad . \quad (3.)$$
$$54x - 24y = 84 \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Adding (3) and (4), member to member (Axiom 1),

$$82x = 164.$$

Here, y has been eliminated by addition.

Again, multiplying both members of (1) by 9, and of (2) by 7,

$$63x + 54y = 180 \quad . \quad . \quad . \quad . \quad (5.)$$
$$63x - 28y = 98 \quad . \quad . \quad . \quad . \quad (6.)$$

Subtracting (6) from (5), member from member, (Axiom 2),

$$82y = 82.$$

Here, x has been eliminated by subtraction.

In the same way, an unknown quantity may be eliminated from any two simultaneous equations. Hence, the

RULE.

Prepare the equations, so that the coefficients of the quantity to be eliminated shall be numerically equal in both; if the signs are unlike, add the equations, member to member; if alike, subtract them, member from member.

In *preparing* the equations above, we multiplied both members of each, by the coefficient of the quantity to be eliminated in the other. They may be prepared in other ways. A better way, in most cases, is to find the least common multiple of the coefficients of the quantity to be eliminated; then multiply both members of each equation by the quotient of this least common multiple by the coefficient of the quantity to be eliminated in that equation. In the first case considered, the least common multiple of 4 and 6 is 12; we might have multiplied both members of (1) by $\frac{12}{6}$, or 2, and of (2) by $\frac{12}{4}$, or 3, giving,

$$14x + 12y = 40$$
$$27x - 12y = 42.$$

Whence, by addition,

$$41x = 82.$$

Here, y has been eliminated as before, but we have a simpler equation.

2. By Substitution.

92. Take the same equations as before:

$$7x + 6y = 20 \quad . \quad . \quad . \quad . \quad . \quad (1.)$$
$$9x - 4y = 14 \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Finding, from (1), the value of y in terms of x,

$$y = \frac{20 - 7x}{6}.$$

Substituting this value for y, in (2),

$$9x - 4\left(\frac{20 - 7x}{6}\right) = 14.$$

Here, y has been eliminated by substitution.

In the same way, we may eliminate an unknown quantity between any two equations. Hence, the

RULE.

Find from one of the equations the value of the quantity to be eliminated; substitute this value for that quantity in the other equation.

3. By Comparison.

93. Take the same equations as before:

$$7x + 6y = 20 \quad . \quad . \quad . \quad . \quad . \quad (1.)$$
$$9x - 4y = 14 \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Finding the values of x in terms of y, from each of the equations,

$$x = \frac{20 - 6y}{7};$$

$$x = \frac{14 + 4y}{9}.$$

Placing these two values equal to each other,

$$\frac{20 - 6y}{7} = \frac{14 + 4y}{9}.$$

Here, x has been eliminated by comparison.

In the same way, an unknown quantity may be eliminated between any two equations. Hence, the

RULE.

Find from each equation the value of the quantity to be eliminated; place these values equal to each other.

Of the three rules given, either one can be used, as may be most convenient. As a general rule, that one will be employed which gives rise to the simplest equations.

SOLUTION OF GROUPS OF SIMULTANEOUS EQUATIONS.

94. Take the three equations,

$$3x + 4y - 2z = 10 \quad . \quad . \quad . \quad . \quad (1.)$$
$$5x - 2y + 3z = 16 \quad . \quad . \quad . \quad . \quad (2.)$$
$$4x + 2y + 2z = 22 \quad . \quad . \quad . \quad . \quad (3.)$$

Combining (1) and (2), also (1) and (3), eliminating z in each case, we have the new group,

$$19x + 8y = 62 \quad . \quad . \quad . \quad . \quad (4.)$$
$$7x + 6y = 32 \quad . \quad . \quad . \quad . \quad (5.)$$

Combining (4) and (5), eliminating y, we have the single equation,

$$29x = 58$$
$$\therefore\ x = 2.$$

Substituting this value of x in (5), we have,

$$14 + 6y = 32$$
$$\therefore\ y = 3.$$

Substituting these values of x and y in (1), we have,

$$6 + 12 - 2z = 10$$
$$\therefore\ z = 4.$$

In the same way, any group of simultaneous equations may be solved. Hence, the

RULE.

I. *Combine one equation of the group with each of the others, eliminating one unknown quantity: there will result a new group containing one equation less than the original group.*

II. *Combine one equation of this group with each of the others, eliminating a second unknown quantity: there will result a new group containing two equations less than the original group.*

III. *Continue the operation until a single equation is found, containing but one unknown quantity.*

IV. *Find the value of this unknown quantity by the preceding rules; substitute this in either one of the group of two equations, and find the value of a second unknown quantity; substitute these in either of the*

group of three, finding a third unknown quantity; and so on, till the values of all are found.

In making the combinations, care should be taken to make them in such a way as to obtain as simple equations as possible. When any unknown quantity does not enter all of the equations, it will in general be best to eliminate that quantity first.

EXAMPLES.

1. Solve the equations,

$$\frac{x}{3} + \frac{y}{5} = 5 \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$2x + \frac{y}{3} = 17 \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Clearing of fractions,

$$5x + 3y = 75 \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$6x + y = 51 \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Combining (3) and (4),

$$13x = 78$$

$$\therefore x = 6$$

Substituting in (4),

$$36 + y = 51$$

$$\therefore y = 15.$$

2. Solve the equations,

$$3x - 4y + 5z = 14 \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$3y + 2z = 10 \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$$12x - 8y - z = 30 \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Combining (1) and (3),

$$8y - 21z = -26 \quad . \quad . \quad . \quad . \quad (4.)$$

Combining (2) and (4),

$$79z = 158$$
$$\therefore z = 2$$

And by successive substitution,

$$y = 2 \text{ and } x = 4.$$

Solve the following groups of simultaneous equations:

3. $\left.\begin{aligned} 3x + 4y &= 18 \\ 2x - y &= 1 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 2. \\ y = 3. \end{cases}$

4. $\left.\begin{aligned} 7x - 3y &= 12 \\ 2x + 2y &= 12 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 3. \\ y = 3. \end{cases}$

5. $\left.\begin{aligned} 5x + 3y &= 26 \\ 5x - y &= 18 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 4. \\ y = 2. \end{cases}$

6. $\left.\begin{aligned} 4x + 3y &= 16 \\ 3x + 4y &= 19 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 1. \\ y = 4. \end{cases}$

7. $\left.\begin{aligned} 6x + y &= 12 \\ x + 6y &= 37 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 1. \\ y = 6. \end{cases}$

8. $\left.\begin{aligned} 4x + 5y &= 17 \\ 3y - 2x &= 8 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = \frac{1}{2}. \\ y = 3. \end{cases}$

9. $\left.\begin{aligned} x - 3y &= 6 \\ 2x + 9y &= 17 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 7. \\ y = \frac{1}{3}. \end{cases}$

10. $\left.\begin{aligned} x + y + z &= 6 \\ 5x + 2y - 3z &= 0 \\ 2x + y - z &= 1 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 1. \\ y = 2. \\ z = 3. \end{cases}$

11. $\left.\begin{aligned} 2x - \tfrac{3}{4}y &= 9 \\ x + y &= 21 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 9. \\ y = 12. \end{cases}$

12. $\left.\begin{aligned} 3x + 2y - z &= 7 \\ x + y - z &= 1 \\ x + 2y + 3z &= 15 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 2. \\ y = 2. \\ z = 3. \end{cases}$

13. $\left.\begin{aligned} \frac{x}{2} - y &= 1 \\ x - \frac{y}{2} &= 8 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 10. \\ y = 4. \end{cases}$

14. $\left.\begin{aligned} \frac{x+y}{10} + \frac{x-y}{2} &= 0 \\ \frac{x+y}{5} + \frac{x-y}{2} &= 1 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 4. \\ y = 6. \end{cases}$

15. $\left.\begin{aligned} \frac{2x-y}{4} - \frac{3}{2} &= \frac{3y}{4} - x - 2 \\ \frac{x+y}{3} &= 2\tfrac{2}{3} \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 3. \\ y = 5. \end{cases}$

16. $\left.\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 12 \\ \frac{x}{3} + \frac{y}{2} &= 13 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 12. \\ y = 18. \end{cases}$

17. $\left.\begin{aligned} 2x - 2y + 3z &= 16 \\ 3x + 5y - 2z &= 6 \\ 4x + 3y - 4z &= -1 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 3. \\ y = 1. \\ z = 4. \end{cases}$

18. $\left.\begin{aligned} \frac{x}{3} + \frac{y}{4} + \frac{z}{5} &= 47 \\ \frac{x}{4} + \frac{y}{5} + \frac{z}{6} &= 38 \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} &= 62 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 24. \\ y = 60. \\ z = 120. \end{cases}$

19. $$\left.\begin{aligned}\frac{x+y}{z} &= 5\\ \frac{y-z}{x} &= 1\\ \frac{x-2}{y} &= \tfrac{1}{3}\end{aligned}\right\}$$ *Ans.* $$\begin{cases} x = 4.\\ y = 6.\\ z = 2.\end{cases}$$

20. $$\left.\begin{aligned}3.4x - .02y &= .01\\ 2x + .\ 4y &= 1.2\end{aligned}\right\}$$ *Ans.* $$\begin{cases} x = .02.\\ y = 2.9.\end{cases}$$

21. $$\left.\begin{aligned}\frac{2x+y}{9} + \frac{7x+6y+11}{18} &= \frac{68-4x}{6}\\ \frac{21}{20}\left(\frac{x}{7} + \frac{y}{4} + \frac{4}{3}\right) &= 4x - \frac{y}{8} - 24\end{aligned}\right\}$$ *Ans.* $$\begin{cases} x = 7.\\ y = 4.\end{cases}$$

22. $$\left.\begin{aligned}\frac{x}{a} + \frac{y}{b} &= 1 - \frac{x}{c}\\ \frac{y}{a} + \frac{x}{b} &= 1 + \frac{y}{c}\end{aligned}\right\}$$ *Ans.* $$\begin{cases} x = \dfrac{(ab-ac-bc)abc}{a^2b^2+a^2c^2-b^2c^2}.\\ y = \dfrac{(ac-ab-bc)abc}{a^2b^2+a^2c^2-b^2c^2}.\end{cases}$$

23. $$\left.\begin{aligned}x + \tfrac{1}{2}(y+z) &= 102\\ y + \tfrac{1}{3}(x+z) &= 78\\ z + \tfrac{1}{4}(x+y) &= 61\end{aligned}\right\}$$ *Ans.* $$\begin{cases} x = 62.\\ y = 46.\\ z = 34.\end{cases}$$

24. $$\left.\begin{aligned}\frac{6y-4x}{3z-7} &= 1\\ \frac{5z-x}{2y-3z} &= 1\\ \frac{y-2z}{3y-2x} &= 1\end{aligned}\right\}$$ *Ans.* $$\begin{cases} x = 10.\\ y = 7.\\ z = 8.\end{cases}$$

25. $$\left.\begin{aligned}\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} &= \frac{38}{5}\\ \frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} &= \frac{61}{6}\\ \frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} &= \frac{161}{10}\end{aligned}\right\}$$ *Ans.* $$\begin{cases} x = \tfrac{1}{2}.\\ y = \tfrac{1}{3}.\\ z = \tfrac{1}{4}.\end{cases}$$

Example 25 may be solved, most readily, by assuming

$$x' = \frac{1}{x}, \quad y' = \frac{1}{y}, \quad \text{and} \quad z' = \frac{1}{z},$$

which gives for the group, 25.

$$\left.\begin{aligned} 3x' - \frac{4y'}{5} + z' &= \frac{38}{5} \\ \frac{x'}{3} + \frac{y'}{2} + 2z' &= \frac{61}{6} \\ \frac{4x'}{5} - \frac{y'}{2} + 4z' &= \frac{161}{10} \end{aligned}\right\} \quad \therefore \quad \left\{\begin{aligned} x' &= 2. \\ y' &= 3. \\ z' &= 4. \end{aligned}\right.$$

From which the values of x, y, and z, may readily be found.

26. $$\left.\begin{aligned} \frac{2}{x} - \frac{5}{3y} + \frac{1}{z} &= \frac{85}{27} \\ \frac{1}{4x} + \frac{1}{y} + \frac{2}{z} &= \frac{443}{72} \\ \frac{5}{6x} - \frac{1}{y} + \frac{4}{z} &= \frac{433}{36} \end{aligned}\right\} \qquad \textit{Ans.} \quad \left\{\begin{aligned} x &= 6. \\ y &= 9. \\ z &= \tfrac{1}{3}. \end{aligned}\right.$$

PROBLEMS.

1. Find two numbers whose sum is a, and whose difference is b.

Let x denote the greater number,
and y denote the lesser number.
From the conditions of the problem, we have,

$$\left.\begin{aligned} x + y &= a \\ x - y &= b \end{aligned}\right\}$$

which are the equations of the problem.

Solving, by the preceding rules, we have,

$$x = \frac{a}{2} + \frac{b}{2}, \text{ and } y = \frac{a}{2} - \frac{b}{2}.$$

Since a and b may be any numbers whatever, we have these general principles by means of which all similar cases can be solved:

1. *The greater number is equal to the half sum increased by the half difference.*

2. *The lesser number is equal to the half sum diminished by the half difference.*

2. If 2 be added to the numerator of a certain fraction, its value will become $\frac{3}{4}$; but if 2 be added to the denominator, its value will be $\frac{1}{2}$. What is the fraction?

Let x denote the numerator,

and y denote the denominator.

From the given conditions, we have the equations of the problem,

$$\frac{x+2}{y} = \frac{3}{4}, \text{ and } \frac{x}{y+2} = \frac{1}{2};$$

whence, $x = 7$, and $y = 12$: Hence, the fraction is $\frac{7}{12}$.

3. The hands of a clock are together at 12 o'clock. When are they next together?

Let x denote the number of minute spaces passed over by the minute hand, and y denote the number of minute spaces passed over by the hour hand.

From the nature of the problem, we have,

$$x = y + 60$$
$$x = 12y.$$

$$\therefore x = 65\tfrac{5}{11}, \quad y = 5\tfrac{5}{11}.$$

Hence, they are together at 1 h. $5\frac{5}{11}$ m.

This problem has been solved already by means of a single unknown quantity, many of the following problems can also be thus solved.

4. A person has 22000 dollars at interest, which yields him 1220 dollars annually; a part bears interest at 5 per cent., and the remainder at 6 per cent. How many dollars in each part?

Let x denote the number of dollars in the first part, and y denote the number of dollars in the second part.

From the conditions,

$$x + y = 22000$$
$$x \times \tfrac{5}{100} + y \times \tfrac{6}{100} = 1220.$$
$$\therefore x = 10000, \quad y = 12000.$$

5. *A's* age is equal to twice *B's* age; 20 years ago, *A's* age was 4 times *B's* age. What are their ages?

Ans. *A's* 60; *B's* 30.

6. There are two numbers: the first added to half the second gives 35; the second added to half the first gives 40. What are the numbers? *Ans.* 20 and 30.

7. A man has three sons; the sum of the ages of the first and second is 27, that of the first and third is 29, and that of the second and third is 32. What are the ages of each? *Ans.* 12, 15, and 17.

8. Two men are in trade; the stock of the first increased by one third that of the second, is \$1700; the stock of the second increased by one fourth that of the first, is \$1800. What is the stock of each?

Ans. \$1200 and \$1500.

9. Find two numbers such that $\frac{1}{2}$ the first plus $\frac{1}{3}$ the second shall equal 45, and $\frac{1}{2}$ the second plus $\frac{1}{3}$ of the first shall equal 40. *Ans.* 50 and 60.

10. The sum of the first and second of three numbers is 13, that of the first and third 16, and that of the second and third 19. What are the numbers?

Ans. 5, 8, and 11.

11. Bought 100 lbs. of sugar and 80 lbs. of coffee for $28, and afterwards bought at the same rates 200 lbs. of sugar and 60 lbs. of coffee for $36. What did each cost per pound?

Ans. Sugar 12 cents, and coffee 20 cents.

12. There are three numbers; the first increased by twice the second and three times the third, makes 74; the second, increased by twice the third and three times the first, makes 90; the third, increased by twice the first and three times the second, makes 100. What are the numbers? *Ans.* 20, 18, and 6.

13. A butcher bought of one person 12 sheep and 20 lambs for 44 dollars, and of a second person 7 sheep and 13 lambs for 27 dollars, at the same rates. How many dollars did he give a piece?

Ans. 2 for sheep, and 1 for lambs

14. Divide the number 1152 into three parts such, that 9 times the sum of the first and second shall be equal to 7 times the sum of the second and third; and if 8 times the first be subtracted from 8 times the second, the remainder shall be equal to the sum of the first and third. *Ans.* 288, 384, and 480.

15. A farmer mixed rye and oats, forming 100 bushels

of the mixture. The rye was worth 96 cents per bushel, the oats 56 cents, and the mixture 72 cents. How many bushels did he use of each?

Ans. 40 of rye and 60 of wheat.

16. A person has two sorts of wine, one worth 40 cents a quart, and the other 24 cents. How much of each kind must he use to form a gallon worth 112 cents.

Ans. 1 quart of the first, 3 quarts of the second.

17. *A* and *B* trade on a joint stock of 833 dollars, and clear 153 dollars. *A's* share of the gain is 45 dollars more than *B's*. What share of the capital did each possess? *Ans.* *A*, \$539; *B*, \$294.

18. Two laborers, *A* and *B*, received 51 dollars. *A* had been employed 14 days, and *B* 15 days; *A* received for 6 days' labor 1 dollar more than *B* got for 4 days' labor. How many dollars did each receive per day? *Ans.* *A*, $1\frac{1}{2}$; *B*, 2.

19. In 80 pounds of an alloy of copper and tin, there are 7 lbs. of copper to 3 of tin. How much copper must be added to the alloy, that there may be 11 lbs. of copper to 4 of tin. *Ans.* 10 lbs.

20. In 3 battalions there are 1905 troops; $\frac{1}{2}$ the number in the first, together with $\frac{1}{3}$ the number in the second, is 60 less than the number in the third; $\frac{1}{2}$ the number in the third, together with $\frac{1}{3}$ the number in the first, is 165 less than the number of the second. How many are there in each battalion?

Ans. 630, 675 and 600.

21. A grocer has three kinds of tea: 12 lbs. of the first, 13 of the second, and 14 of the third are together worth 25 dollars: 10 of the first, 17 of the

second, and 11 of the third are together worth 24 dollars; 6 of the first, 12 of the second, and 6 of the third are together worth 15 dollars. What is the value of a pound of each?

Ans. 50 cents, 60 cents, and 80 cents.

22. A owes \$1200, and B \$2500; but neither has money enough to pay his debts. Says A to B, "Lend me $\frac{1}{8}$ of your fortune, and I can pay my debts;" says B to A, "Lend me $\frac{1}{5}$ of your fortune, and I can pay mine." What fortune had each?

Ans. B, had \$2400; and A, \$900.

23. The united ages of a father and son are 80 years; and if the age of the son be doubled, it will exceed the father's age by 10 years. What is the age of each?

Ans. 50 and 30.

24. A travels uniformly along a certain road, B starts an hour afterwards in pursuit, and after 4 hours finds by inquiry that he is travelling $1\frac{1}{2}$ miles per hour slower than A; he then doubles his rate of travel, and overtakes A, $6\frac{1}{3}$ hours from the time he started in pursuit. At what rate did A travel, and what was the rate that B travelled at first?

Ans. A's rate, $9\frac{3}{4}$ miles; B's, $8\frac{1}{4}$.

25. There are 32 gallons of wine in two casks. If from the first there be drawn into the second as much as there is in the second; then if there be drawn from the second into the first as much as remains in the first; and then if there be drawn from the first into the second as much as remains in the second, there will be 16 gallons in each cask. How many gallons were there originally in each? *Ans.* 22 and 10.

26. A cistern can be filled by 3 pipes. The first can fill it in 4 hours, the first and second together can fill it in 3 hours, and the third can fill it in 2 hours. How long will it take for them all to fill it together, and how long will it take the second alone to fill it?

Ans. All in 1 h. 12 m. The second in 12 hours.

27. A cistern has two discharge cocks: they both run together for two hours when the first one is closed; the second one then empties it in 2 hours and 48 minutes. Had the second one been closed at the end of two hours, the first one would have emptied it in 4 hours and 40 minutes. In what time could each empty it alone?

Ans. The first in 10 hours; the second in 6 hours.

DISCUSSION OF PROBLEMS.

95. The DISCUSSION of a problem consists in making every possible supposition upon the *arbitrary* quantities which enter it, and interpreting the results.

An ARBITRARY QUANTITY is a quantity to which a value may be given at pleasure.

The discussion of the following problem indicates the general method of discussion, as well as illustrates the meaning of the *plus* and *minus* signs, as signs of *interpretation*, together with the ideas to be attached to the symbols, 0, ∞, and $\frac{0}{0}$.

PROBLEM OF THE COURIERS.

96. Two couriers, A and B, travel along the same line, $R'\ R$, in the same direction, $R'\ R$, and at uniform rates; the courier A travels m miles per hour, and the

courier B, n miles per hour. Now, supposing them to be separated by a distance a at any epoch, say 12 o'clock, when are they together?

R' A B R

Let the position of the rearmost courier, A, be taken as the origin of distances, and suppose all distances estimated towards B to be positive.

Denote the number of hours from the epoch to the time they are together by t. Denote the distance the courier B travels in the time t, by x; then will the distance that the courier A travels, in the same time, be denoted by $a + x$.

Then, since the distance travelled is equal to the number of hours multiplied by the rate per hour, we have the equations:

$$mt = a + x,$$
$$nt = x.$$

Whence, by solving,

$$t = \frac{a}{m-n}, \quad x = \frac{na}{m-n}, \quad \text{and} \quad x + a = \frac{ma}{m-n}.$$

To discuss these values. The distance between the couriers may be assumed at pleasure; hence, a is arbitrary. The rates of travel may, also, be assumed at pleasure; hence, m and n are arbitrary. From the nature of the case, a can never be negative; hence, the only suppositions that can be made upon a, are that it is positive, or $a > 0$; or that it is equal to zero, or $a = 0$. The only hypotheses that can be made upon m and n, are $m > n$, $m < n$, and $m = n$. By combining these suppositions, we may obtain the six suppositions:

1. $a > 0$, and $m > n$.
2. $a > 0$, and $m < n$.
3. $a > 0$, and $m = n$.
4. $a = 0$, and $m > n$.
5. $a = 0$, and $m < n$.
6. $a = 0$, and $m = n$.

We shall make each hypothesis separately.

1. $a > 0$, and $m > n$.

This hypothesis makes both terms of the fraction $\frac{a}{m-n}$ positive; hence, the value of t is positive.

We interpret this result as showing that the time required is *after* the epoch 12 o'clock.

This interpretation is in accordance with the supposition made; for, if $m > n$, the courier A, travels faster than the courier B; hence, after the epoch, the distance between them continually diminishes, and, consequently, at some time after the epoch they must come together.

The same hypothesis makes both x and $x + a$ positive, showing, as before, that they are together somewhere to the right of the origin of distances.

2. $a > 0$, and $m < n$.

This hypothesis makes the numerator of the fraction $\frac{a}{m-n}$ positive, and the denominator negative; hence, the value of t is *negative.*

We interpret this result as showing that the time when they are together is *before* the epoch 12 o'clock.

This interpretation is also in accordance with the supposition made; for, if $m < n$, the courier B travels faster than the courier A; hence, after the epoch, they will continually separate. Before the epoch, they must have been together at some time, *previous* to which B was approaching A, and after which the two were continually separating.

The same hypothesis makes both x and $x + a$ negative, showing as before, that the point where they were together is on the left of the origin of distances.

From this discussion, we see that if we regard time *after* an epoch, as positive, time *before* that epoch must be regarded as negative; if we regard distance, in *one direction* from a point as positive, distance in the *opposite direction* must be regarded as negative.

It may be shown, generally, that if we agree to consider quantity, in any sense, as positive, quantity in an opposite sense *must* be regarded as negative. As signs of interpretation, + and − are diametrically opposed to each other.

3. $a > 0$, and $m = n$.

This hypothesis makes the numerator of the fraction $\frac{a}{m-n}$ finite, and the denominator 0; hence, the value of t is equal to a finite quantity divided by 0, or ∞ (Art. 73).

This result is interpreted as showing, that the time from the epoch when they are together, is greater than any assignable time; that is, they are never together.

This interpretation is also in accordance with the supposition made; for, if $m = n$, the two couriers travel at the same rate, and as they are separated by a distance a, at the epoch, they will always *have been*, and *will always be*, at the same distance apart; that is, they can never be together.

The same supposition makes both x and $x + a$ infinite, showing that the distance from the origin to the point where they are together, is greater than any assignable distance.

4 and 5. $a = 0$, $m > n$, and $a = 0$, $m < n$.

Either of these hypotheses makes the numerator of the fraction $\frac{a}{m-n}$ zero, and the denominator finite; hence, the value of t is equal to 0 divided by a finite quantity, or 0 (Art. 72).

This result is interpreted as showing that they are together at the epoch.

This interpretation is also in accordance with the supposition made; for, since $a = 0$, they are together at the epoch; and since m and n are unequal, they travel at different rates; consequently, they can never be together after the epoch, nor could they ever have been together before the epoch.

The same hypothesis makes both x and $x + a$ equal to 0, showing that the distance from the origin to the point where they are together, is 0.

6. $a = 0$, and $m = n$.

This hypothesis makes both terms of the fraction $\frac{a}{m-n}$ equal to 0; hence, the value of t is $\frac{0}{0}$, or, indeterminate (Art. 74).

This result we interpret as showing that there are an *infinite number of times* when they are together.

This interpretation is also in accordance with the supposition made; for, a being 0, they are together at the epoch; and m being equal to n, they travel at the same rate; consequently, they will be together at all times, both before and after the epoch 12 o'clock.

The same hypothesis renders the values of x and $x + a$ indeterminate, showing that there are an infinite number of distances, from the origin of distance to the points where they are together.

97. The general solution, $t = \frac{a}{m-n}$, may be applied to the solution of a great number of similar problems. The general rule being, that the time required, in hours, is equal *to the initial distance of separation* divided by the *relative velocity.*

We shall only apply it to a single case, *the clock problem.*

To find when the hands of a clock will be together between 1 and 2 o'clock. Here, 12 o'clock is the origin of distance, and if we take the minute space on the dial as 1, the *initial distance*, that is, the distance to be *gained*, is 60; the rate of the minute hand is 60, that of the hour hand, 5: hence,

$$t = \frac{60}{60-5} = 1\tfrac{1}{11} \text{ hours, or 1 h. } 5\tfrac{5}{11} \text{ min.}$$

To find when the hands are together between 2 and 3 o'clock, we have the initial space, 2×60, or 120, and the rates as before. Hence,

$$t = \frac{120}{60-5} = 2\tfrac{2}{11} \text{ hours, or 2 h. } 10\tfrac{10}{11} \text{ min.}$$

To find when they are together between 3 and 4 o'clock, 4 and 5 o'clock, &c., the initial spaces are 180, 240, &c. Hence,

$$t = \frac{180}{55} = 3\tfrac{3}{11} \text{ hours, equal to 3 h. } 16\tfrac{4}{11} \text{ min.}$$

$$t = \frac{240}{55} = 4\tfrac{4}{11} \text{ hours, equal to 4 h. } 21\tfrac{9}{11} \text{ min., \&c.}$$

CHAPTER VI.

FORMATION OF POWERS; BINOMIAL FORMULA.

98. A Power of a quantity is the product obtained by taking that quantity any number of times as a factor.

If the quantity be taken once as a factor, we have the first power, the other factor being 1, the base of all numbers; if taken twice, we have the second power; if three times, the third power; if n times, the n^{th} power, n being any whole number whatever.

A power is indicated by means of the exponential sign: thus,

$(5a)^2$, signifying the square of $5a$;

$(3a + b)^3$, signifying the third power of $3a + b$;

$(7a - 2x)^n$, signifying the n^{th} power of $7a - 2x$.

By analogy, any quantity written with a negative or fractional exponent, as $(2a - x)^{-3}$, $(7a - 2b)^{\frac{2}{3}}$, &c., is called a power, and read, $(2a - x)$ to the minus three power, or $(7a - 2b)$ to the two-third power.

POWERS OF MONOMIALS.

99. Let it be required to find the third power, or cube, of $7a^2x$.

From the definition of a power, and the rule for Multiplication, we have,

$$(7a^2x)^3 = 7a^2x \times 7a^2x \times 7a^2x = 7 \times 7 \times 7a^2a^2a^2xxx = 343a^6x^3.$$

In a similar manner, any monomial may be raised to any power. Hence, the

RULE.

Raise the coefficient to the required power for a new coefficient; write after this all the letters, giving to each an exponent equal to the product of its original exponent by the exponent of the power.

If the given monomial is positive, all of its powers are positive; if it is negative, its square is positive, its cube negative, its fourth power positive, and so on. In general, any even power of a negative quantity is positive, and every odd power negative. These principles follow from the *rule for signs*, in Multiplication.

EXAMPLES.

1. $(3ax^2y)^2$. *Ans.* $9a^2x^4y^2$.
2. $(2a^2yx^3)^3$. *Ans.* $8a^6y^3x^9$.
3. $(-2axy^2)^3$. *Ans.* $-8a^3x^3y^6$.
4. $(-3a^2bc^3x)^4$. *Ans.* $81a^8b^4c^{12}x^4$.
5. $(-7dx^3y^2)^3$. *Ans.* $-343d^3x^9y^6$.
6. $(2x^3yz)^5$. *Ans.* $32x^{15}y^5z^5$.
7. $(-d^2x^3y^4)^3$ *Ans.* $-d^6x^9y^{12}$.
8. $(-x^2y^3z^4)^4$. *Ans.* $x^8y^{12}z^{16}$.
9. $(4axy^3z)^3$. *Ans.* $64a^3x^3y^9z^3$.
10. $(-3a^2y^3)^4$. *Ans.* $81a^8y^{12}$.

POWERS OF FRACTIONS.

100. Let it be required to find the third power of $\frac{2a^2x}{3by}$.

From the definition of a power, and the rule for multiplication of fractions, we have,

$$\left(\frac{2a^2x}{3by}\right)^3 = \frac{2a^2x}{3by} \times \frac{2a^2x}{3by} \times \frac{2a^2x}{3by} = \frac{8a^6x^3}{27b^3y^3};$$

and similarly for other fractions. Hence, the

RULE.

Raise the numerator to the required power for a new numerator, and the denominator to the required power for a new denominator.

The rule for signs is the same as in the last article.

EXAMPLES.

1. $\left(\frac{a}{b}\right)^2$. *Ans.* $\frac{a^2}{b^2}$.

2. $\left(\frac{ax}{by}\right)^3$. *Ans.* $\frac{a^3x^3}{b^3y^3}$

3. $\left(\frac{-2ax}{3y}\right)^4$. *Ans.* $\frac{16a^4x^4}{81y^4}$.

4. $\left(\frac{-3y}{4x}\right)^3$. *Ans.* $\frac{-27y^3}{64x^3}$.

5. $\left(\frac{2ax^2y}{3bc^2}\right)^2$. *Ans.* $\frac{4a^2x^4y^2}{9b^2c^4}$.

6. $\left(-\frac{dx}{3y^2}\right)^3$. *Ans.* $-\frac{d^3x^3}{27y^6}$.

7. $\left(\frac{axy^3}{2bz^2}\right)^3$. *Ans.* $\frac{a^3x^3y^9}{8b^3z^6}$.

8. $\left(-\frac{3ay^4}{2b^2x}\right)^4$. *Ans.* $\frac{81a^4y^{16}}{16b^8x^4}$.

9. $\left(-\frac{2x^2y^3}{abc}\right)^2$. *Ans.* $\frac{4x^4y^6}{a^2b^2c^2}$.

10. $\left(-\frac{2ab^2c^3x^4}{3}\right)^6$. *Ans.* $\frac{64a^6b^{12}c^{18}x^{24}}{729}$.

101. The rule for raising a monomial to any power holds true when the exponents of any of the letters are negative, or when the exponent of the required power is negative.

Let it be required to find the square of $3a^{-2}x^{-4}$, and that power of $2ax^2$, whose exponent is -3.

It has been shown that any factor may be changed from the denominator to the numerator, or from the numerator to the denominator, by changing the sign of its exponent (Art. 32). Hence,

$$(3a^{-2}x^{-4})^2 = \left(\frac{3}{a^2x^4}\right)^2 = \frac{9}{a^4x^8};$$

$$\text{also,} \quad (2ax^2)^{-3} = \frac{1}{(2ax^2)^3} = \frac{1}{2^3a^3x^6}.$$

Transferring the factors to the numerator, we have,

$$(3a^{-2}x^{-4})^2 = 9a^{-4}x^{-8};$$

$$\text{also,} \quad (2ax^2)^{-3} = 2^{-3}a^{-3}x^{-6} = \tfrac{1}{8}a^{-3}x^{-6}.$$

In the same way, the truth of the principle may be shown in all similar cases.

EXAMPLES.

1. $(a^{-2})^2$. *Ans.* a^{-4}.

2. $(x^{-3}y)^{-2}$. *Ans.* x^6y^{-2}.

3. $(2x^2y^3)^{-2}$. *Ans.* $\frac{1}{4}x^{-4}y^{-6}$.

4. $(2x^{-2}y^{-3})^{-2}$. *Ans.* $\frac{1}{4}x^4y^6$.

5. $(ax^2y^3z^{-2})^{-3}$. *Ans.* $a^{-3}x^{-6}y^{-9}z^6$.

6. $(2a^{-3}b^{-2}c^3)^3$. *Ans.* $8a^{-9}b^{-6}c^9$.

7. $(-3x^{-1}y^{-2})^3$. *Ans.* $-27x^{-3}y^{-6}$.

8. $(5a^{-2}b^{-3}c^{-2})^{-3}$. *Ans.* $\frac{1}{125}a^6b^9c^6$.

9. $(-2x^2y^{-3})^{-3}$. *Ans.* $-\frac{1}{8}x^{-6}y^9$.

10. $(-2x^{-2}y^{-3})^4$. *Ans.* $16x^{-8}y^{-12}$.

11. $(-3ax^2y^{-1}z^{-2})^{-2}$. *Ans.* $\frac{1}{9}a^{-2}x^{-4}y^2z^4$.

12. $(-3x^{-3}y^4)^{-4}$. *Ans.* $\frac{1}{81}x^{12}y^{-16}$.

POWERS OF BINOMIALS. BINOMIAL FORMULA.

102. A binomial may be raised to any power by the process of continued multiplication, but when the exponent of the power is greater than 2, the operation is greatly abridged by making use of the *binomial formula.*

103. The BINOMIAL FORMULA, is a formula by means of which a binomial may be raised to any power, without going through the process of continued multiplication.

104. The following powers of $x + y$ are found by actual multiplication:

$$(x+y)^1 = x + y.$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

And in the same way, the higher powers might be obtained. If we examine the powers already deduced, we see that they are all formed according to the following laws:

1. Law of Exponents.

The exponent of the leading letter in the first term is equal the exponent of the power, and the exponent of that letter goes on diminishing by 1 *in each term towards the right till the last term, where it is* 0*: the exponent of the following letter is* 0 *in the first term, and the exponent of that letter goes on increasing by* 1 *in each term towards the right to the last term, where it is equal to the exponent of the power.*

2. Law of Coefficients.

The coefficient of the first term is 1*; the coefficient of any succeeding term is found by multiplying the coefficient of the preceding term by the exponent of the leading letter in that term, and dividing the product by the number of terms preceding the required term.*

Let us assume that these laws of formation hold true for a power whose exponent is m, any positive whole number. The application of these laws gives,

$$(x+y)^m = x^m + mx^{m-1}y + m \cdot \frac{m-1}{2}x^{m-2}y^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}x^{m-3}y^3 + \&c., + y^m.$$

If both members of this equation be multiplied by $(x+y)$, the first member will become $(x+y)^{m+1}$; to find what the second member becomes, let us perform the multiplication.

$$x^m + mx^{m-1}y + m \cdot \frac{m-1}{2} x^{m-2}y^2 + \&c. + y^m$$
$$x + y$$

$$x^{m+1} + mx^m y + m \cdot \frac{m-1}{2} x^{m-1}y^2 + \&c. + xy^m$$
$$x^m y + mx^{m-1}y^2 + \&c. + mxy^m + y^{m+1}$$

$$x^{m+1} + \begin{matrix} m \\ +1 \end{matrix} \left| \begin{matrix} x^m y + \\ +m \end{matrix} \right. m \cdot \frac{m-1}{2} \left| \begin{matrix} x^{m-1}y^2 + \&c. + \\ \\ \end{matrix} \right. \begin{matrix} m \\ +1 \end{matrix} \left| \begin{matrix} xy^m + y^{m+1} \\ \\ \end{matrix} \right.$$

But,

$$m \cdot \frac{m-1}{2} + m = m\left(\frac{m-1}{2} + 1\right) = \frac{(m+1)m}{1.2}.$$

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} + m \cdot \frac{m-1}{2} = m \cdot \frac{m-1}{2}\left(\frac{m-2}{3} + 1\right)$$
$$= \frac{(m+1)\,m(m-1)}{1.2.3}.$$

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}$$
$$= \frac{(m+1)\,m(m-1)\,(m-2)}{1.2.3.4}.$$

The law of the coefficients is evident.

Substituting these, in the product, we have,

$$(x+y)^{m+1} = x^{m+1} + (m+1)x^m y + \frac{(m+1)m}{1.2} x^{m-1}y^2$$
$$+ \frac{(m+1).m(m-1)}{1.2.3} \cdot x^{m-2}y^3 +, \&c., + y^{m+1}.$$

If we examine the $(m+1)$ power, we see that the assumed laws of formation hold good in it. Hence, we have proved, that *if the assumed laws of formation hold good when the exponent of the power is m, they will also hold good when the exponent is* $(m+1)$.

Now, we have proved by actual multiplication, that the assumed laws hold good when the exponent is 5; hence, from what we have just proved, they will hold good when the exponent is 6. Hence, from the principle demonstrated, they must hold good when the exponent is 7; and if for 7, then for 8; if for 8, then for 9: and so on, by successive deduction, it may be shown that the laws hold good for any whole number whatever.

If then, we denote any whole number whatever by n, we shall have, from the preceding demonstration,

$$(x+y)^n = x^n + nx^{n-1}y + n \cdot \frac{n-1}{2}x^{n-2}y^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}x^{n-3}y^3 + \&c. + y^n,$$

which is the *binomial formula.*

We have only proved the truth of the formula when n is a positive whole number; it is, however, true when n is either positive or negative, entire or fractional, as will be demonstrated in the Appendix.

If we now change the places of x and y, we shall have, by the laws of formation,

$$(y+x)^n = y^n + ny^{n-1}x + n \cdot \frac{n-1}{2} \cdot y^{n-2}x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}y^{n-3}x^3 + \&c. + x^n.$$

The second member of this equation is the same as

the second member of formula already deduced, but taken in a reverse order. Comparing the two, we see that the coefficients of the two taken in the same order are equal, which shows that *the coefficients of the second member of the binomial formula, taken at equal distances from the extremes, are equal.* Hence, in forming any power of a binomial, it is only necessary to find the coefficients to the middle of the development; the remaining ones can be written from these in a reverse order.

From the laws of formation, we see that there will be $n+1$ terms in the development of the n^{th} power of a binomial. Hence, when n is an odd number, we must find $\frac{n+1}{2}$ coefficients, but when n is an even number, we must find $\frac{n+2}{2}$ coefficients.

To apply the formula to find any power of a binomial, we raise the first term to its successive powers as high as the n^{th}, and substitute them for the corresponding powers of x in the formula; we then raise the second term to its successive powers up to the n^{th}, and substitute them in the formula for the corresponding powers of y, substituting for n the exponent of the powers.

EXAMPLES.

1. Find the third power of $a+b$.

Here, a takes the place of x, b the place of y, and 3 the place of n; making these substitutions in the formula, we have,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

2. Find the fourth power of $c+d$.

Here, c takes the place of x, d the place of y, and 4 the place of n; making these substitutions, we have,

$$(c + d)^4 = c^4 + 4c^3d + 6c^2d^2 + 4cd^3 + d^4.$$

3. Find the fifth power of $a + b$.

We may write the literal parts of each term of the development by the *law of exponents*, giving,

$$a^5,\ a^4b,\ a^3b^2,\ a^2b^3,\ ab^4, \text{ and } b^5.$$

The coefficients may be formed by the *law of coefficients*. The coefficient of the first term is 1, that of the second is 5, that of the third term is $\frac{5 \times 4}{2}$, or 10, and the remaining coefficients are the same, but taken in the reverse order. Hence,

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

4. Find the sixth power of $(a + b)$.

Proceeding as in the preceding example, we have,

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

5. Find the fifth power of $c + d$.

Ans. $c^5 + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + d^5$.

6. Find the sixth power of $c + d$.

Ans. $c^6 + 6c^5d + 15c^4d^2 + 20c^3d^3 + 15c^2d^4 + 6cd^5 + d^6$.

7. Find the third power of $a - b$.

Here, a takes the place of x in the formula, $-b$ the place of y, and 3 the place of n, giving,

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

When the second term of the binomial is negative, all the odd terms are positive, whilst all the even terms are negative.

8. Find the fourth power of $c - d$.

Ans. $c^4 - 4c^3d + 6c^2d^2 - 4cd^3 + d^4$.

9. Find the fifth power of $a - b$.

Ans. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

10. Find the sixth power of $c - d$.

Ans. $c^6 - 6c^5d + 15c^4d^2 - 20c^3d^3 + 15c^2d^4 - 6cd^5 + d^6$.

It is to be observed, that any power of the difference of two quantities may be written out by the two laws for exponents and coefficients, provided the signs of the terms be made alternately *plus* and *minus*, as in the above examples.

105. When the terms of the binomial have coefficients, we may still write out any power of it by means of the binomial formula.

11. Find the cube of $2a + 3b$.

Here, $2a$ takes the place of x in the formula, $3b$ the place of y, and 3 the place of n, giving,

$$(2a+3b)^3 = (2a)^3+3(2a)^2(3b)+3(2a)(3b)^2+(3b)^3 \ldots (1.);$$

or, performing the operations indicated,

$$(2a + 3b)^3 = 8a^3 + 36a^2b + 54ab^2 + 27b^3.$$

If we examine the second member of equation (1), we see that each term is made up of three factors; 1st, the numerical factor; 2d, some power of $2a$; and 3d, some power of $3b$. The powers of $2a$ are arranged in descending order towards the right, the last term involving the 0 power of $2a$ or 1; the powers of $3b$ are arranged in ascending order from the first term where the 0 power enters to the last term.

The operation of raising a binomial involving coeffi-

cients, is most readily effected by writing the three factors of each term in a vertical column, and then performing the multiplications as indicated below.

$$\begin{array}{llllll} 1 & + 3 & + 3 & + 1 & \text{Coefficients.} \\ 8a^3 & + 4a^2 & + 2a & + 1 & \text{Powers of } 2a. \\ 1 & + 3b & + 9b^2 & + 27b^3 & \text{Powers of } 3b. \end{array}$$

$$(2a + 3b)^3 = 8a^3 + 36a^2b + 54ab^2 + 27b^3$$

The preceding operation hardly requires explanation. In the first line, write the numerical coefficients corresponding to the particular power; in the second line, write the descending powers of the leading term to the 0 power; in the third line, write the ascending powers of the following term from the 0 power upwards. It will be easiest to commence the second line on the right hand. The multiplication should be performed from above, downwards.

12. Find the cube of $3a + 2b$.

OPERATION.

$$\begin{array}{llllll} 1 & + 3 & + 3 & + 1 & \text{Coefficients.} \\ 27a^3 & + 9a^2 & + 3a & + 1 & \text{Powers of } 3a. \\ 1 & + 2b & + 4b^2 & + 8b^3 & \text{Powers of } 2b. \end{array}$$

$$(3a + 2b)^3 = 27a^3 + 54a^2b + 36ab^2 + 8b^3$$

13. Find the square of $7x - 3y$.

OPERATION.

$$\begin{array}{llll} 1 & + 2 & + 1 & \text{Coefficients.} \\ 49x^2 & + 7x & + 1 & \text{Powers of } 7x. \\ 1 & - 3y & + 9y^2 & \text{Powers of } -3y \end{array}$$

$$\therefore (7x - 3y)^2 = 49x^2 - 42xy + 9y^2$$

14. Find the cube of $2x - 3y$.

OPERATION.

$$\begin{array}{llll} 1 & +3 & +3 & +1 \quad \text{Coefficients.} \\ 8x^3 & +4x^2 & +2x & +1 \quad \text{Powers of } 2x. \\ 1 & -3y & +9y^2 & -27y^3 \quad \text{Powers of } -3y. \end{array}$$

$$(2x-3y)^3 = 8x^3 - 36x^2y + 54xy^2 - 27y^3$$

15. Find the cube of $\frac{1}{2}x + \frac{1}{3}y$.

OPERATION.

$$\begin{array}{llll} 1 & +3 & +3 & +1 \quad \text{Coefficients.} \\ \frac{1}{8}x^3 & +\frac{1}{4}x^2 & +\frac{1}{2}x & +1 \quad \text{Powers of } \frac{1}{2}x. \\ 1 & +\frac{1}{3}y & +\frac{1}{9}y^2 & +\frac{1}{27}y^3 \quad \text{Powers of } \frac{1}{3}y. \end{array}$$

$$(\tfrac{1}{2}x + \tfrac{1}{3}y)^3 = \tfrac{1}{8}x^3 + \tfrac{1}{4}x^2y + \tfrac{1}{6}xy^2 + \tfrac{1}{27}y^2$$

16. Find the fourth power of $\frac{1}{2}a - 3b$.

OPERATION.

$$\begin{array}{llll} 1 & +4 & +6 & +4 \quad +1 \\ \frac{1}{16}a^4 & +\frac{1}{8}a^3 & +\frac{1}{4}a^2 & +\frac{1}{2}a \quad +1 \\ 1 & -3b & +9b^2 & -27b^3 \quad +81b^4 \end{array}$$

$$(\tfrac{1}{2}a-3b)^4 = \tfrac{1}{16}a^4 - \tfrac{3}{2}a^3b + \tfrac{27}{2}a^2b^2 - 54ab^3 + 81b^4$$

17. Find the cube of $2ax - 3by^2$.

OPERATION.

$$\begin{array}{llll} 1 & +3 & +3 & +1 \\ 8a^3x^3 & +4a^2x^2 & +2ax & +1 \\ 1 & -3by^2 & +9b^2y^4 & -27b^3y^6. \end{array}$$

$$(2ax - 3by^2)^3 = 8a^3x^3 - 36a^2bx^2y^2 + 54ab^2xy^4 - 27b^3y^6.$$

18. Find the fourth power of $\frac{ax}{b} - \frac{cy}{d}$.

OPERATION.

$$\begin{array}{lllll} 1 & +4 & +6 & +4 & +1 \\ \frac{a^4x^4}{b^4} & +\frac{a^3x^3}{b^3} & +\frac{a^2x^2}{b^2} & +\frac{ax}{b} & +1 \\ 1 & -\frac{cy}{d} & +\frac{c^2y^2}{d^2} & -\frac{c^3y^3}{d^3} & +\frac{c^4y^4}{d^4} \end{array}$$

$$\left(\frac{ax}{b}-\frac{cy}{d}\right)^4 = \frac{a^4x^4}{b^4} - \frac{4a^3cx^3y}{b^3d} + \frac{6a^2c^2x^2y^2}{b^2d^2} - \frac{4ac^3xy^3}{bd^3} + \frac{c^4y^4}{d^4}.$$

19. Find the fourth power of $mx + ny$.

Ans. $m^4x^4 + 4m^3nx^3y + 6m^2n^2x^2y^2 + 4mn^3xy^3 + n^4y^4$.

20. Raise $a - 2x$ to the fourth power.

Ans. $a^4 - 8a^3x + 24a^2x^2 - 32ax^3 + 16x^4$.

21. $\left(x+\frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$.

22. $\left(\frac{x}{a^2}-\frac{a^2}{x}\right)^3 = \frac{x^3}{a^6} - \frac{3x}{a^2} + \frac{3a^2}{x} - \frac{a^6}{x^3}$.

23. $(x^{pq}-1)^3 = x^{3pq} - 3x^{2pq} + 3x^{pq} - 1$.

24. $(e^x - e^{-x})^3 = e^{3x} - e^{-3x} - 3(e^x - e^{-x})$.

25. $(5-4x)^4 = 625 - 2000x + 2400x^2 - 1280x^3 + 256x^4$

26. $(5a^2c^2d - 4abd^2)^4 = 625a^8c^8d^4 - 2000a^7bc^6d^5 + 2400a^6b^2c^4d^6 - 1280a^5b^3c^2d^7 + 256a^4b^4d^8$.

POWERS OF POLYNOMIALS.

106. The polynomial $a + b + c$, may be written under the form, $a+(b+c)$; hence, $(a+b+c)^n = [a+(b+c)]^n$; also, $(2a-x+3y+4z)^n$, equal to $[(2a-x)+(3y+4z)]^n$;

and so on, for polynomials containing any number of terms.

To raise any polynomial to a power, we write it under the form of a binomial, each term of which may be a binomial, or some other polynomial; we then form the powers of these parts (by known methods), and proceed exactly as with a true binomial.

EXAMPLES.

1. Find the cube of $a + (b + c)$.

OPERATION.

$$\begin{array}{l} 1 + 3 + 3 + 1 \\ a^3 + a^2 + a \quad\quad 1 \\ 1 + (b+c) + (b^2+2bc+c^2) + (b^3+3b^2c+3bc^2+c^3) \end{array}$$

$$(a+b+c)^3 = a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3$$

2. Find the square of $(2a - x) + (3y + 4z)$.

OPERATION.

$$\begin{array}{c} 1 \quad + 2 + 1 \\ (4a^2-4ax+x^2)+(2a-x)+1 \\ 1 \quad +(3y+4z)+(9y^2+24yz+16z^2) \end{array}$$

$$(2a-x+3y+4z)^2 = 4a^2-4ax+x^2+12ay+16az-6xy-8xz+9y^2+24yz+16z^2$$

If a polynomial contains 5 terms, we first divide it into two parts, a binomial and a trinomial; find the powers of the trinomial, as shown above, and then proceed as indicated, and similarly for any polynomial whatever.

3. $(1 - 2x + 3x^2)^3 = 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

4. $(a^2 - a + \frac{1}{4})^2 = a^4 - 2a^3 + \frac{3}{2}a^2 - \frac{1}{2}a + \frac{1}{16}$.

5. $\left(x - \frac{1}{x} - 1\right)^3 = x^3 - \frac{1}{x^3} - 3x^2 - \frac{3}{x^2} + 5$.

6. $\left(7x^2 - \frac{x}{5} + 3\right)^2 = 49x^4 - \frac{14}{5}x^3 + \frac{1051}{25}x^2 - \frac{6}{5}x + 9$

7. $\left(x + \frac{a}{3} - \frac{b}{2}\right)^2 = x^2 + \frac{2}{3}ax - bx - \frac{1}{3}ab + \frac{b^2}{4} + \frac{a^2}{9}$

8. $\left(x^2 - x + \frac{1}{4}\right)^2 = x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{x}{2} + \frac{1}{16}$.

9. $(3a^2 - 2ab + 5b^2)^2 = 9a^4 - 12a^3b + 34a^2b^2 - 20ab^3 + 25b^4$.

10. $(2x^2 - 3x + 4)^2 = 4x^4 - 12x^3 + 25x^2 - 24x + 16$.

11. $(x^3 + 2x^2 + 3x + 4)^2 = x^6 + 4x^5 + 10x^4 + 20x^3 - 25x^2 + 24x + 16$.

CHAPTER VII.

EXTRACTION OF ROOTS.

107. A Root of a quantity is one of its equal factors. If a quantity be resolved into 2 equal factors, one of them is called the *square root:* the symbol for the square root is $\sqrt{\ }$. If a quantity be resolved into 3 equal factors, one of these factors is called the *cube root:* the symbol for the cube root is $\sqrt[3]{\ }$. If a quantity be resolved into n equal factors, one of these factors is called the n^{th} *root:* the symbol for the n^{th} root is $\sqrt[n]{\ }$.

108. The operation of finding one of the n equal factors of a quantity (n being any positive whole number whatever), is called *extracting its* n^{th} *root.*

Instead of employing the symbols $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[n]{\ }$, to indicate the *square*, *cube*, n^{th} roots, it is more convenient to employ the fractional exponents, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{n}$, which indicate the same thing. Thus,

$$\sqrt{a} = a^{\frac{1}{2}}, \quad \sqrt[3]{a} = a^{\frac{1}{3}}, \quad \sqrt[4]{a} = a^{\frac{1}{4}} \quad \ldots \quad \sqrt[n]{a} = a^{\frac{1}{n}}.$$

It will be shown hereafter, that quantities affected with fractional exponents may be operated upon by the same rules as when they are affected with entire exponents.

SQUARE ROOT OF NUMBERS.

109. The Square root of a number is one of its two equal factors. Thus, $25 = 5 \times 5$; hence, 5 is the square root of 25; that is, $\sqrt{25} = 5$, or $(25)^{\frac{1}{2}} = 5$.

The following table, verified by actual multiplication, is employed in finding the square root of any number less than 100.

TABLE.

1	4	9	16	25	36	49	64	81	100.	Powers.
1	2	3	4	5	6	7	8	9	10.	Roots.

To employ the table for finding the square root of a number less than 100. *Look for the number in the first line; if it is found there, its square root will be found immediately under it; if it is not found there, it will fall between two numbers in that line, and its square root will be found between the two numbers immediately below; the lesser of the two will be the entire part of the root, and will be the true root to within less than* 1.

If a number is greater than 100, it square root will be greater than 10, that is, it will contain *tens* and *units*. Let N denote such a number, x the tens of its square root, and y the units; then will

$$N = (x + y)^2 = x^2 + 2xy + y^2 = x^2 + (2x + y)y.$$

That is, the number is equal to the *square of the tens* in its root, plus *twice the product of the tens by the units*, plus *the square of the units*.

We first find the *tens* of the root. Since the square of tens can contain no significant figure less than *hundreds*, the two figures on the right may be pointed off,

and the square of the tens will be found in the number to the left of the point. Having found the tens, we subtract their square from the given number, and the remainder will be made up of twice the product of the tens by the units plus the square of the units. To find the units, divide the remainder by twice the tens, and the quotient will be the units, or a number greater than the units. To test it, add it to twice the tens, and multiply the sum by the number found; if the product is equal to the remainder, or less than it, the number found is the root sought; but if greater, diminish it by 1, and test as before, till the correct number is found.

In dividing the remainder by twice the tens, the last figure may be omitted, since twice the product of the tens by the units can never give a significant figure in the units' place.

EXAMPLE.

Find the square root of 1764.

OPERATION.

Pointing off the two right hand figures, there remains the number 17, the greatest perfect square in which is 16. The square root of 16, or 4, is therefore the number of *tens* of the required root.

```
17 64 | 42
16
---
82 | 16 4
     164
     ---
       0
```

Write this on the right, after the manner of a quotient. Subtracting 16 hundreds (the square of 4 tens) from the number, the remainder is 164. Doubling the tens, we have 8, which is contained in 16 4, exclusive of the right hand figure, 2 times, which is the number of units: adding this to the 8 tens (which is done by writing it after 8), and multiplying the sum by 2, we

have 164, equal to the remainder. Hence, if we write the 2 after the 4, in the root, we shall have the root sought equal to 42.

When the number contains more than 4 places of figures, we point off two figures from the right, as before. The operation is then reduced to finding the square root of the remaining numbers, that is, the tens of the root. In finding this root, for the same reason as before, we point off another period of two figures, and the operation then is reduced to finding the square root of the remaining number, that is, the *tens of the tens*, or hundreds of the root. If this number contains more than two places of figures, for the same reason as before, we point off another period of two figures to the right, when the square root of the remaining number will give the *tens of hundreds*, or *thousands of the root;* and so we continue pointing off, till a number is found on the left containing less than three places of figures: the operation is then but a successive application of that already explained. Hence, the

RULE.

I. *Point the number off into periods of two figures each, beginning at the units' place.*

II. *Find the greatest perfect square in the first period on the left, and place its square root on the right, after the manner of a quotient in a division; then subtract the square of this number from the first period, and bring down the next period for a remainder.*

III. *Double the root already found, and see how often it is contained in this remainder, exclusive of the right hand figure; write this quotient for a second figure of*

the root, annex it also to the divisor used: multiply the divisor thus increased by the quotient figure found, subtract this product from the first remainder, and bring down the next period for a second remainder.

IV. *Double the root already found and proceed as before, continuing the operation till every period has been employed. If the final remainder is* 0, *the root is exact, if it is not* 0, *the root found is true to within less than* 1.

EXAMPLES.

1. Find the square root of 273529.

OPERATION.

$$\begin{array}{r|l} 27\,35\,29 & 523 \\ 25 & \\ \hline 102 \mid 235 & \\ 204 & \\ \hline 1043 \mid 3129 & \\ 3129 & \\ \hline 0. & \end{array}$$

2. Find the square root of 61009. *Ans.* 247.

Find the square roots of the following numbers:

3. 4096.	*Ans.* 64.	7. 68492176.	*Ans.* 8276.
4. 582169.	*Ans.* 763.	8. 1018081.	*Ans.* 1009.
5. 956484.	*Ans.* 978.	9. 9803161.	*Ans.* 3131.
6. 57198969.	*Ans.* 7563.	10. 1522756.	*Ans.* 1234.

SQUARE ROOT OF FRACTIONS.

110. It has been shown (Art. 100) that a fraction is squared by squaring both numerator and denominator separately: reversing the process, we have for extracting the square root of a fraction the following

RULE.

Extract the square root of the numerator for a new numerator, and the square root of the denominator for a new denominator.

EXAMPLES.

Extract the square roots of the following fractions:

1. $\frac{25}{36}$. *Ans.* $\frac{5}{6}$.

2. $\frac{256}{625}$. *Ans.* $\frac{16}{25}$.

3. $\frac{98}{242}$, or $\frac{49}{121}$. *Ans.* $\frac{7}{11}$.

4. $\frac{392}{18}$, or $\frac{196}{9}$. *Ans.* $\frac{14}{3}$.

5. $\frac{2209}{196}$. *Ans.* $\frac{47}{14}$.

6. $54\frac{12}{25}$, or $\frac{1369}{25}$. *Ans.* $7\frac{2}{5}$.

111. When the terms of the fraction, after being reduced to its simplest form, are not perfect squares, the exact root is then impossible. In such cases, we may multiply both terms by any number which will make the denominator a perfect square. Then, extracting the square root of the numerator to the nearest unit for a new numerator, and the square root of the denominator for a new denominator, the resulting fraction will be the true root to within less than the fractional unit.

EXAMPLE.

Find the square roots of the following fractions, approximately:

1. $\frac{19}{8} = \frac{38}{16}$ *Ans.* $\frac{6}{4}$ to within $\frac{1}{4}$.

2. $\frac{19}{8} = \frac{19 \times 32}{8 \times 32}$. *Ans.* $\frac{24}{16}$ to within $\frac{1}{16}$.

By increasing the factor introduced in both terms, we may make the resulting root true to any degree of exactness.

3. $\frac{38}{5} = \frac{190}{25}$. *Ans.* $\frac{14}{5}$ to within $\frac{1}{5}$.

4. $\frac{45}{7} = \frac{45 \times 343}{7 \times 343}$. *Ans.* $\frac{124}{49}$ to within $\frac{1}{49}$.

5. $\frac{57}{8} = \frac{57 \times 20000}{8 \times 20000}$. *Ans.* $\frac{1019}{400}$ to within $\frac{1}{400}$.

6. $\frac{21}{2} = \frac{4200}{400}$. *Ans.* $\frac{65}{20}$ to within $\frac{1}{20}$.

SQUARE ROOT OF DECIMALS.

112. The denominator of a decimal fraction will be a perfect square when the number of decimal places is even, and the number of decimal places in the root of the decimal will be half the number of decimal places in the given decimal. Hence, from the preceding principle, we have the following rule for finding the square root of any decimal fraction, to any degree of exactness.

RULE.

Annex 0's *to the decimal till the number of decimal places is twice the number of decimal places required*

in the root; extract the square root of the result as though it were a whole number, and point off the required number of decimal places in the root.

If we have a vulgar fraction, it may be converted into an equivalent decimal fraction, and then the above rule is applicable. The rule is also applicable to whole numbers.

EXAMPLES.

Find the square roots of the following decimals, approximately:

1. 9.6 = 9.600000. *Ans.* 3.098 to within .001.
2. 7.65 = 7.65000000. *Ans.* 2.7658 to within .0001.
3. 15.2379. *Ans.* 3.90357 to within .00001.
4. $\frac{4}{7}$, or .571428. *Ans.* 0.755 to within .001.
5. $\frac{1}{11}$. *Ans.* 0.24253 to within .00001.
6. $10\frac{3}{10}$. *Ans.* 3.209 to within .001.
7. 5, or 5.000000. *Ans.* 2.236 to within .001.
8. 22. *Ans.* 4.69 to within .01.
9. 153. *Ans.* 12.36931 to within .00001.
10. 101. *Ans.* 10.04987 to within .00001.

113. It is to be observed, that if a whole number is not a perfect square its exact square root cannot be found even decimally. That is, *the square root of a whole number can never be a fraction.* To prove this principle, let n represent any whole number, and suppose that its square root is an irreducible fraction, $\frac{a}{b}$, that is,

$$\sqrt{n} = \frac{a}{b}:$$

Squaring both members (Axiom 5),

$$n = \frac{a^2}{b^2} = \frac{a \times a}{b \times b}.$$

Since $\frac{a}{b}$ is irreducible, there are no common factors in a and b; and because the square of $\frac{a}{b}$ is formed by taking each factor of a twice for a new numerator, and each factor of b twice for a denominator, it follows that a^2 and b^2 have no common factors; or, in other words, $\frac{a^2}{b^2}$ is an irreducible fraction. We have, therefore, a whole number n, equal to an irreducible fraction, $\frac{a^2}{b^2}$, which is impossible; therefore, the supposition that the square root of a whole number can be an irreducible fraction, is absurd. This kind of demonstration is called "*Reductio ad absurdum.*"

CUBE ROOT OF NUMBERS.

114. The cube root of a number is one of its three equal factors. Thus, $27 = 3 \times 3 \times 3$; hence, 3 is the cube root of 27; also, 4 is the cube root of 64, because $4 \times 4 \times 4 = 64$.

The following table, verified by actual multiplication, is employed in finding the cube root of any number less than 1000:

TABLE.

1	8	27	64	125	216	343	512	729	1000
1	2	3	4	5	6	7	8	9	10.

To employ the table in finding the cube root of a number less than 1000. *Look for the number in the first line, if found there, its cube root is immediately*

below it; if the number is not in the first line, it will be between two numbers in that line, and its root will fall between the corresponding numbers in the second line; the lesser of the two is the entire part of the cube root required, that is, it is the cube root to within less than 1.

If the number is greater than 1000, its cube root will be greater than 10, that is, it will be made up of *tens* and *units*. Denoting the number by N, the tens in its cube root by a, and the units by b, we shall have,

$$N = (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

That is, a number is equal to *cube of the tens* of its cube root, plus *three times the product of the square of the tens by the units*, plus *three times the product of the tens by the square of the units*, plus *the cube of the units*.

We first find the *tens* of the root. Since the cube of the tens can contain no significant figure of a less denomination than thousands, we point off three figures from the right hand, and find the cube root of the remaining number to within less than 1: this will be the tens required. We next subtract the cube of the tens from the period on the left, and bring down the period on the right for a remainder. This remainder is made up of 3 times the square of the tens by the units, plus other parts. If, therefore, we divide it by 3 times the square of the tens, the quotient will be the units or some greater number. Since 3 times the square of the tens by the units can give no significant figure less than hundreds, in making the division, the last two figures in the remainder may be pointed off and not used. To test the accuracy of the *units* found, annex it to the tens and

form the cube of the result; if this cube is equal to, or less than the given number, the root found is correct; if it is greater, the last figure must be diminished until the cube of the root found is equal to or less than the given number.

If the number to the left of the first period of three figures on the right contains more than three figures, the tens of the root will be made up of *tens of tens*, or *hundreds*, and *units of tens*, or *tens;* and for the same reason as before, a second period of three figures must be pointed off, and so on until the first period on the left contains three figures or less; it may contain but 2 figures, or it may contain but 1. The root is then found by a repetition of the process above given. Hence, the

RULE.

I. *Separate the number into periods of three figures each, beginning at the right hand; the left hand period may contain less than three figures.*

II. *Find the greatest perfect cube in the first period on the left, and write its cube root on the right, after the manner of a quotient in division. Subtract the cube of the root found from the first period on the left, and bring down the next period for a remainder.*

III. *Divide the remainder, exclusive of the two right hand figures, by three times the square of the root found, and annex the quotient to the root. Cube this number (diminishing it by* 1, 2, *&c., if necessary), and subtract the result from the first two periods, and bring down the next period for a second remainder.*

IV. *Divide this remainder by three times the square of the root already found, proceeding as before, and con-*

tinue the operation till all the periods have been used. If the final remainder is 0, *the root is exact; if not, the root is true to within less than* 1.

EXAMPLE.

1. Find the cube root of 389017.

Here we have two periods. The greatest cube in 389 is 343 (Table) and its cube root is 7. Subtracting 343 from 389, and bringing down 017, pointing off two figures, we have 460, which contains $3 \times 7^2 = 147$, 3 times.

OPERATION.

$$\begin{array}{rl} 389\,017 & \mid 73 \\ 343 & \\ \hline 147 \mid & 460\,17 \\ \hline (73)^3 = & 389\,017 \\ & 0. \end{array}$$

Annexing 3 to 7, we have 73; and cubing, we find the given number, which being subtracted, leaves for a remainder 0.

2. Find the cube root of 111980168.

OPERATION.

$$\begin{array}{rl} 111\,980\,168 & \mid 482 \\ 64 & (49)^3 = 117649. \\ \hline 48 \mid & 479\,80 \\ \hline (48)^3 = & 110\,592 \\ \hline 6912 \mid & 13881\,68 \\ \hline & 1119801\,68 \\ \hline & 0 \end{array}$$

Here we have three periods. The greatest cube in the first is 64, and its root is 4. Subtracting 64 from 111, bringing down the second period and pointing off two figures, we have 479, which contains $3 \times 4^2 = 48$, 9 times. Annexing 9 to 4 and cubing 49, we find 117649.

which is greater than the first two periods. Cubing 48, we find a result less than the first two periods; hence, 8 is the second figure of the root. Subtracting $(48)^3$ from the first two periods, bringing down the third period and pointing off two figures, we have 13881, which contains $3 \times (48)^2 = 6912$, 2 times. Annexing 2 to 48 and cubing, we find for a result the given number: hence, the cube root required is 482.

3. Find the cube root of 2460375. *Ans.* 135.
4. Find the cube root of 11089567. *Ans.* 223.
5. Find the cube root of 40353607. *Ans.* 343.
6. Find the cube root of 403583419. *Ans.* 739.
7. Find the cube root of 115501303. *Ans.* 487.

115. The process of extracting the cube root of a number may be generalized. Let n be the index of the *root*, n being any whole number. We point off the given number into periods of n figures each, beginning at the right: the left hand period may contain less than n figures. We next find the greatest n^{th} power in the first period on the left, and write its n^{th} root for the first figure of the required root. We then subtract the n^{th} power of this root from the first period, bring down the next period, and point off $n - 1$ figures to the right. Dividing the number thus obtained by n times the $(n-1)^{\text{th}}$ power of the root found, we get the second figure of the root, or a figure too great. The figure found is tested, and the proof continued as in finding the cube root.

EXAMPLE.

1. To find the 5th root of 115856001.

OPERATION.

$$\begin{array}{rl} & 1158\,56001 \mid \underline{41} \\ & \underline{1024} \\ 5 \times (4)^4 = 1280 \;) & \overline{1345\,6001} \\ (41)^5 = & \underline{11585\,6001} \\ & 0 \end{array}$$

116. When the index of the required root is composed of two factors, the operation of extracting the root may be simplified.

Let N be any number, and assume

$$\sqrt[m]{\sqrt[n]{N}} = r \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Raising both members of (1) to the m^{th} power, we have,

$$\sqrt[n]{N} = r^m \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Raising both members of (2) to the n^{th} power, we have,

$$N = r^{mn} \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Extracting the mn^{th} root of both members of (3), we have,

$$\sqrt[mn]{N} = r \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Things which are equal to the same thing are equal to each other; hence, placing the first members of (1) and (4) equal, we have,

$$\sqrt[mn]{N} = \sqrt[m]{\sqrt[n]{N}}.$$

From which, we conclude that *the* mn^{th} *root of any number is equal to the* m^{th} *root of the* n^{th} *root of that number.*

We may, therefore, factor the index and extract that root of the number indicated by one of the factors, and

then that root of the result indicated by the other factor. It will be simplest to begin with the least factor.

EXAMPLES.

1. Find the fourth root of 923521.

We have, $\sqrt{923521} = 961$,

and, $\sqrt{961} = 31$.

Hence, 31 is the root required.

2. Find the sixth root of 191102976.

We have, $\sqrt{191102976} = 13824$,

and, $\sqrt[3]{13824} = 24$. *Ans.*

3. Find the fourth root of 65536. *Ans.* 16.

HIGHER ROOTS OF FRACTIONS.

117. It has been shown (Art. 100), that a fraction may be raised to any power by raising the numerator to that power for a new numerator, and the denominator to that power for a new denominator. Reversing the principle, we have the rule for finding any root of a fraction.

RULE.

Extract the required root of the numerator for a new numerator, and the same root of the denominator for a new denominator.

EXAMPLES.

1. Find the cube root of $\frac{8}{27}$. *Ans.* $\frac{2}{3}$.
2. Find the cube root of $\frac{64}{125}$. *Ans.* $\frac{4}{5}$.

3. Find the cube root of $1\frac{3}{8}\frac{4}{2}\frac{9}{7}$. *Ans.* $\frac{1}{2}\frac{1}{4}$.
4. Find the fourth root of $29\frac{8}{1}\frac{3}{7}$. *Ans.* $2\frac{1}{3}$.
5. Find the fourth root of $104\frac{4}{6}\frac{3}{2}\frac{4}{5}$. *Ans.* $3\frac{1}{5}$.
6. Find the sixth root of $11\frac{2}{6}\frac{5}{4}$. *Ans.* $1\frac{1}{4}$.

118. If the terms of the fraction are not exact powers of the degree indicated, we may multiply both terms by such a number as will make the denominator an exact power of that degree. Then, extracting the required root of the resulting numerator to within less than 1, and writing the result over the required root of the denominator of the fraction, the result will be the true root, to within less than the fractional unit.

EXAMPLES.

Find the cube roots of the following numbers:

1. $\frac{173}{32} = \frac{346}{64}$. *Ans.* $\frac{7}{4}$ to within $\frac{1}{4}$.

2. $\frac{125}{256} = \frac{250}{512}$. *Ans.* $\frac{6}{8}$ to within $\frac{1}{8}$.

3. $278 = \frac{278000}{1000}$. *Ans.* $\frac{65}{10}$, or 6.5 to within .1.

4. Find the fourth root of $\frac{210}{2187} = \frac{630}{6561}$.

Ans. $\frac{5}{9}$ to within $\frac{1}{9}$.

119. The denominator of a decimal fraction is a perfect n^{th} power (n being any whole number), when the number of decimal places is divisible by n, and the number of decimal places in its n^{th} root is the n^{th} part of the number of decimal places in the given deci-

mal. Hence, from the preceding principle, we have the following rule for finding the n^{th} root of any decimal fraction to any desired degree of accuracy.

RULE.

Annex 0's *to the given decimal, till the number of decimal places is n times the number in the required root; extract the* n^{th} *root of the result, as though it were a whole number, and point off the required number of decimal places in the root.*

The rule is applicable when the fraction is a vulgar one, for we may convert it into a decimal one by known rules; it is also applicable to finding an approximate root of a whole number. It may be shown, as in square root, that the n^{th} root of a whole number can never be a fraction; hence, if the number is not a perfect n^{th} power, its n^{th} root can never be exactly found.

EXAMPLES.

Find the cube roots of the following numbers, approximately:

1. 5.8. *Ans.* 1.7967 to within .0001.
2. 102.875. *Ans.* 4.6856 to within .0001.
3. $\frac{2}{3}$. *Ans.* 0.873 to within .001.
4. $\frac{4}{6}$. *Ans.* 0.941 to within .001.
5. 82. *Ans.* 4.344 to within .001.
6. 550. *Ans.* 8.198 to within .001.
7. Find the fourth root of 72.
Ans. 2.91 to within .01.

8. Find the sixth root of 28.25.

Ans. 1.745 to within .001.

Find the fourth roots of the following numbers:

9. $\frac{7}{4}$. *Ans.* 1.15 to within .01.

10. $8\frac{14}{15}$. *Ans.* 1.69 to within .01.

11. 13. *Ans.* 1.89 to within .01.

12. Find the cube root of 58230.6.

Ans. 38.76 to within 01.

ROOTS OF MONOMIALS.

120. We have seen (Art. 99) that any monomial can be raised to any power by raising the coefficient to the required power for a new coefficient, and giving to each letter an exponent equal to its original exponent, multiplied by the exponent of the required power. Reversing the process, we have a rule for extracting any root of a monomial.

RULE.

Extract the required root of the coefficient for a new coefficient; after this, write all the letters, giving to each an exponent equal to its original exponent, divided by the index of the required root.

This rule, combined with the rule for extracting any root of a fraction already given, enables us to extract any root of a monomial, whether entire or fractional.

EXAMPLES.

Find the square roots of the following fractions:

1. $9a^2b^4x^2$. *Ans.* $3ab^2x$.

2. $49a^2x^4y^6$. *Ans.* $7ax^2y^3$

3. $\frac{a^4y^2z^8}{16}$. *Ans.* $\frac{a^2yz^4}{4}$.

4. $\frac{25a^4x^4y^2}{81a^2b^2c^4}$. *Ans.* $\frac{5a^2x^2y}{9abc^2}$.

5. $25a^{-2}b^{-4}c^2$. *Ans.* $5a^{-1}b^{-2}c$.

Find the cube roots of the following fractions:

6. $8a^3b^6y^6$. *Ans.* $2ab^2y^2$.

7. $\frac{8a^3y^3}{27d^6z^6}$. *Ans.* $\frac{2ay}{3d^2z^2}$.

8. Find the fourth root of $\frac{16a^4y^4}{625z^8}$. *Ans.* $\frac{2ay}{5z^2}$.

9. Find the cube root of $343x^{-3}y^{-6}$.

Ans. $7x^{-1}y^{-2}$.

10. Find the fifth root of $x^{-10}y^{-20}z^5$.

Ans. $x^{-2}y^{-4}z$.

Since the square of $+a$ is a^2, and the square of $-a$ is also a^2, it follows that a^2 has two square roots, $+a$ and $-a$. Further, since every even power of a positive quantity is equal to the same power of that quantity taken with a negative sign, it follows that every positive quantity has two square roots, two fourth roots, two sixth roots, &c., which are equal numerically, but have contrary signs. Thus,

$$\sqrt{25a^2b^2} = \pm 5ab\,;\ \sqrt[4]{16a^4b^8} = \pm 2ab^2\,;\ \sqrt[6]{a^{12}b^6} = \pm a^2b.$$

It is impossible that any even power of a quantity, either positive or negative, should be a negative quantity. Hence, it is equally impossible to extract any even root of a negative quantity. An indicated even root of a negative quantity, is called an imaginary quantity. Thus, $\sqrt{-4}$, $\sqrt{-a^2}$, $\sqrt[4]{-b^2}$, are imaginary quantities.

Every odd power of a negative quantity is itself negative, and consequently every odd root of a negative quantity must be negative. Every odd root of a positive quantity must of course be positive. Hence, the

RULE.

Every even root of a positive quantity must have the double sign, $\pm$; *every odd root of any quantity must have the sign of the quantity.*

In the preceding examples, only the numerical value of the roots have been considered; hereafter, the proper signs will be prefixed to the results.

EXAMPLES.

11. Find the cube root of $-27a^3x^{-3}y^6$.

Ans. $-3ax^{-1}y^2$.

12. Find the square root of $\frac{64}{9}a^4b^4$. *Ans.* $\pm\frac{8}{3}a^2b^2$.

13. Find the 4th root of $a^{4m}x^{8m}y^{4m}$.

Ans. $\pm a^m x^{2m}y^m$.

14. Find the cube root of $-512a^{-3}$.

Ans. $-8a^{-1}$.

15. Find the 5th root of $-32x^{-5}y^{-10}$.

Ans. $-2x^{-1}y^{-2}$.

16. Find the square root of $\frac{256a^2b^4}{9x^4y^2}$.

Ans. $\pm\frac{16ab^2}{3x^2y}$.

17. Find the cube root of $\frac{729x^3y^{-3}}{a^6b^{-6}}$.

Ans. $+\frac{9xy^{-1}}{a^2b^{-2}}$.

18. Find the 4th root of $\frac{81a^4x^4}{y^8}$. *Ans.* $\pm\frac{3ax}{y^2}$.

19. Find the 5th root of $-\frac{243x^5}{y^{10}}$. *Ans.* $-\frac{3x}{y^2}$.

20. Find the square root of $900a^2x^4y^6$.
Ans. $\pm 30ax^2y^3$.

ROOTS OF POLYNOMIALS.

121. To deduce a rule for extracting the square root of a polynomial, let us suppose the root to be known, and to be arranged with respect to some letter. We may regard the root as composed of *the first term* plus *the sum of all the other terms.* Hence, its square, which is the given polynomial, will be made up of *the square of the first term*, plus *twice the product of the first term by the sum of all the other terms*, plus *the square of the sum of all the other terms* (Art. 106). Now the square of the first term of the root must be of a higher degree with respect to the leading letter than any other term. Hence, if we arrange the given polynomial with respect to any letter, the square root of the first term will be the first term of the required square root. If the square of this term of the root be subtracted from the given polynomial, and the first term of the remainder be divided by twice the first term of the root, the quotient will be the second term of the root. The remainder of the process of finding the square root, is entirely analogous to that already explained for finding the square root of a whole number, and will be best understood from an example. Let it be required to extract the square root of $9x^4 + 12x^3 + 28x^2 + 16x + 16$.

OPERATION.

$$
\begin{array}{rl}
9x^4 + 12x^3 + 28x^2 + 16x + 16 \;\big|\; 3x^2 + 2x + 4 & \\
9x^4 & \\
\hline
6x^2 + 2x \;\big|\; 12x^3 + 28x^2 + 16x + 16 & \ldots\ 1st\ remainder. \\
12x^3 + 4x^2 & \\
\hline
6x^2 + 4x + 4 \;\big|\; 24x^2 + 16x + 16 & \ldots\ 2d\ remainder. \\
24x^2 + 16x + 16 & \\
\hline
0 & \ldots\ 3d\ remainder.
\end{array}
$$

The polynomial is arranged with respect to x. The square root of the first term is $3x^2$. Subtracting $9x^4$ from the given polynomial, and dividing the first term of the remainder by $6x^2$, which is double $3x^2$, we find $2x$ for the second term, which we add to the root and also to the divisor. Multiplying the divisor thus augmented by $2x$, and subtracting the product from the first remainder, we have a second remainder. Doubling the root already found, and dividing the first term of the second remainder by the first term of the last divisor, we find 4 for the third term. Adding this to the root and to the second divisor, and multiplying the divisor by the last term found, we find for the final remainder 0; hence, the root required is, $3x^2 + 2x + 4$.

In the same way, the square root of any polynomial may be found. Hence, the

RULE.

I. *Arrange the polynomial with reference to some letter, and extract the square root of the first term for the first term of the root. Subtract the square of this term from the polynomial for the first remainder.*

II. *Double the root already found, and place it on the left of the first remainder for a divisor; divide the*

first term of the first remainder by this divisor, for the second term of the root; add the quotient to the root found, and also to the divisor; multiply the divisor, thus augmented, by the last term of the root found, and subtract the product from the first remainder for a second remainder.

III. *Double the root already found for a second divisor. Divide the first term of the second remainder by the first term of the second divisor for the third term of the root; add this term to the root and to the second divisor, and proceed as before, continuing the operation as far as desirable.*

If a final remainder is found, equal to 0, the root is exact; if not, the root is only approximate.

EXAMPLES.

1. Extract the square root of $a^4 - 2a^3 + 3a^2 - 2a + 1$.

OPERATION.

$$\begin{array}{l|l} a^4 - 2a^3 + 3a^2 - 2a + 1 & a^2 - a + 1 \\ a^4 & \\ \hline 2a^2 - a \;|\; -2a^3 + 3a^2 & \\ \quad -2a^3 + a^2 & \\ \hline 2a^2 - 2a + 1 \;|\; 2a^2 - 2a + 1 & \\ \quad 2a^2 - 2a + 1 & \\ \hline \end{array}$$

In finding the several remainders, all of the terms need not be brought down: only as many as are needed.

Extract the square roots of the following polynomials:

2. $9x^2 - 30ax + 25a^2 + 5a^3 + \frac{a^4}{4} - 3a^2x$. *A.* $3x - 5a - \frac{a^2}{2}$.

3. $4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4$.

Ans. $2x^2 + 2ax + 4b^2$.

4. $\frac{a^2}{b^2}+\frac{b^2}{a^2}+2\left(\frac{a}{b}+\frac{b}{a}\right)+3.$ *Ans.* $\frac{a}{b}+\frac{b}{a}+1$

5. $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}+\frac{xy}{3}-\frac{xz}{4}-\frac{yz}{6}.$ *Ans.* $\frac{x}{2}+\frac{y}{3}-\frac{z}{4}$

6. $\frac{x^2}{9}+\frac{4y^2}{25}+\frac{z^2}{16}+\frac{4xy}{15}-\frac{xz}{6}-\frac{yz}{5}.$ *Ans.* $\frac{x}{3}+\frac{2y}{5}-\frac{z}{4}.$

7. $x^4+2px^3+(p^2-2q)x^2-2pqx+q^2.$

Ans. $x^2+px-q.$

8. $(x+x^{-1})^2-4(x-x^{-1}).$ *Ans.* $x-x^{-1}-2.$

9. $9a^{2m}+6a^{3m+1}+25c^{2m-4}-30a^mc^{m-2}+a^{4m+2}-10a^{2m+1}c^{m-2}.$ *Ans.* $3a^m-5c^{m-2}+a^{2m+1}.$

10. $49x^4+9-\frac{14x^3}{5}-\frac{6x}{5}+\frac{1051x^2}{25}.$

Ans. $7x^2-\frac{x}{5}+3.$

11. $1+x.$ *Ans.* $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-$ &c., &c.

12. $\frac{x^2}{y^2}\left(\frac{x^2}{4y^2}+1\right)+\frac{4y^2}{x^2}\left(\frac{y^2}{x^2}+1\right)+3.$

Ans. $\frac{x^2}{2y^2}+\frac{2y^2}{x^2}+1.$

122. The process of finding the cube root, or any higher root of a polynomial, is entirely analogous to that for finding the corresponding root of a whole number. The reasoning in the two cases is entirely the same. The method will be best understood by an example. Let it be required to find the cube root of

$$x^6+6x^5-40x^3+96x-64.$$

OPERATION.

$$
\begin{array}{rl|l}
 & x^6 + 6x^5 - 40x^3 + 96x - 64 & x^2 + 2x - 4 \\
 & x^6 & \\
\hline
3x^4 & + 6x^5 & \\
\hline
 & x^6 + 6x^5 + 12x^4 + 8x^3 & \\
\hline
3x^4 + \&c. & - 12x^4 - 48x^3 & \\
\hline
 & x^6 + 6x^5 - 40x^3 + 96x - 64 & \\
\hline
 & 0 &
\end{array}
$$

The cube root of x^6 is x^2: subtracting $(x^2)^3$, or x^6, from the given polynomial, the first term of the remainder is $6x^5$; the other terms are not brought down. Three times the square of x^2, is contained in $6x^5$, $2x$ times; hence, $2x$ is the second term of the root. Cubing $x^2 + 2x$, and subtracting from the original polynomial, the first term of the remainder is $-12x^4$. If now, we square $x^2 + 2x$, and multiply by 3, the first term is $3x^4$, which is contained in $-12x^4$, -4 times; hence, -4 is the third term of the root. Cubing $x^2 + 2x - 4$, we find the original polynomial; hence, it is the root required.

Again, let it be required to find the fourth root of

$$16x^4 - 128x^3y + 384x^2y^2 - 512xy^3 + 256y^4.$$

Extracting the square root, we have,

$$4x^2 - 16xy + 16y^2;$$

and the square root of this polynomial gives,

$$2x - 4y,$$

which is the fourth root required.

EXAMPLES.

Find the cube roots of the following polynomials:

1. $8x^3 - 12x^2 + 6x - 1$. *Ans.* $2x - 1$.

2. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

Ans. $x^2 - 2x + 1$.

3. $64a^6 - 288a^5 + 1080a^3 - 1458a - 729$.

Ans. $4a^2 - 6a - 9$.

4. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

Ans. $1 - 2x + 3x^2$.

By extracting the required root of the first and last terms, two terms of the root may in general be found, from which the remaining ones may, in general, be determined by inspection; the whole root may then be verified as above. Experience will suggest many other simplifications in what is at best but a tedious operation, and fortunately not of frequent use.

CHAPTER VIII.

TRANSFORMATION OF RADICALS. RADICAL EQUATIONS.

123. A RADICAL is an indicated root of an imperfect power. Thus, $\sqrt{3}$, $\sqrt[3]{4}$, are radicals. A radical is sometimes called a *surd*. Radicals are divided into degrees, the degree being determined by the index of the root. When the root is indicated by a fractional exponent, the degree of the radical is determined by the denominator of the fractional exponent.

$7\sqrt{a}$, $\sqrt{5}$, $b^{\frac{1}{2}}$, $(7ax)^{\frac{1}{2}}$, are *radicals of the second degree.*

$3\sqrt[3]{a}$, $\sqrt[3]{6}$, $x^{\frac{1}{3}}$, $(3xy^2)^{\frac{1}{3}}$, are *radicals of the third degree.*

$\sqrt[n]{c}$, $2\sqrt[n]{2}$, $z^{\frac{1}{n}}$, $(2az^3)^{\frac{1}{n}}$, are *radicals of the n^{th} degree.*

124. Radicals are SIMILAR, when the radical parts are the same; this requires that they be of the *same degree*, and that the quantities under the radical sign be the *same in each.* Thus, $7\sqrt{3}$ and $4\sqrt{3}$, are similar; also $12\sqrt[3]{7}$ and $4\sqrt[3]{7}$; also $ax^{\frac{1}{n}}$ and $bx^{\frac{1}{n}}$.

PRINCIPLES EMPLOYED IN TRANSFORMING RADICALS.

125. Let n denote any whole number whatever, and assume

$$\sqrt[n]{a} \times \sqrt[n]{b} = p \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Raising both members of (1) to the n^{th} power, remembering that $(\sqrt[n]{a})^n = a$, and $(\sqrt[n]{b})^n = b$, we have,

$$ab = p^n \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Extracting the n^{th} root of both members of (2), we have,

$$\sqrt[n]{ab} = p \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Things equal to the same thing are equal to each other: hence, equating the first members of (1) and (3), we have,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} \quad . \quad . \quad . \quad . \quad (4.)$$

Again, assume

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = q \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

Raising both members of (5) to the n^{th} power,

$$\frac{a}{b} = q^n \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

Extracting the n^{th} root of both members of (6),

$$\sqrt[n]{\frac{a}{b}} = q \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

Equating the first members of (5) and (7),

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad . \quad . \quad . \quad . \quad (8.)$$

It may be shown, as in Art. 116, that when m and n are any positive whole numbers, and a any quantity, we have,

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

From equations (4), (8), and (9), we have the following principles:

1. *The product of the n^{th} roots of two quantities, is equal to the n^{th} root of their product, and the reverse.*

2. *The quotient of the n^{th} roots of two quantities, is equal to the n^{th} root of their quotient, and the reverse.*

3. *The m^{th} root of the n^{th} root of any quantity, is equal to the mn^{th} root of that quantity, and the reverse.*

These principles are of continual application in the transformation of radicals.

126. It has already been shown, that radical quantities can be expressed by means of fractional exponents. The following table of equivalent expressions indicates the different methods of expressing radicals, powers, and reciprocals, according to the conventional principles already alluded to:

$\sqrt[3]{x}$,	$\sqrt[n]{y}$,	equivalent to	$x^{\frac{1}{3}}$,	$y^{\frac{1}{n}}$.
$\sqrt[3]{x^2}$,	$\sqrt[n]{y^m}$,	equivalent to	$x^{\frac{2}{3}}$,	$y^{\frac{m}{n}}$.
x^{-3},	y^{-n},	equivalent to	$\frac{1}{x^3}$,	$\frac{1}{y^n}$
$\sqrt[4]{\frac{1}{x^3}}$,	$\sqrt[n]{\frac{1}{x^m}}$,	equivalent to	$x^{-\frac{3}{4}}$,	$x^{-\frac{m}{n}}$.

These are but different methods of writing the same expressions, and we may, at any time, pass from one form of expression to its equivalent.

To reduce a radical to its simplest form.

127. A radical is in its SIMPLEST FORM when there is no factor under the sign which is a perfect power.

Take the radical, $\sqrt{a^3x^4y^3}$. By factoring, we have,

$$\sqrt{a^3x^4y^3} = \sqrt{a^2x^4y^2 \times ay}.$$

Hence, from Principle (1), we have,

$$\sqrt{a^3x^4y^3} = \sqrt{a^2x^4y^2} \times \sqrt{ay} = ax^2y\sqrt{ay}.$$

In a similar manner, other radicals may often be simplified. Hence, the rule for reducing a radical to its simplest form.

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect power indicated. Extract the required root of this factor, and write it as a factor without the radical sign, leaving the other factor under the sign.

Before pronouncing upon the similarity of two radicals they should both be reduced to their simplest form.

EXAMPLES.

Reduce the following radicals to their simplest forms:

1. $\sqrt{48a^4x^2y} = \sqrt{16a^4x^2 \times 3y}$. *Ans.* $4a^2x\sqrt{3y}$.

2. $\sqrt{\frac{50a^2}{147b^2x}} = \sqrt{\frac{25a^2}{49b^2} \times \frac{2}{3x}}$. *Ans.* $\frac{5a}{7b}\sqrt{\frac{2}{3x}}$.

3. $\sqrt{(a+x)(a^2-x^2)} = \sqrt{(a+x)^2(a-x)}$.
Ans. $(a+x)\sqrt{a-x}$.

4. $\sqrt[3]{2a^4x + a^3x^2} = \sqrt[3]{a^3(2ax+x^2)}$.
Ans. $a\sqrt[3]{2ax+x^2}$.

5. $\sqrt[3]{\frac{24}{25}} + \sqrt[3]{\frac{5}{72}} + \sqrt[3]{\frac{192}{125}}$.

Ans. $2\sqrt[3]{\frac{3}{25}} + \frac{1}{2}\sqrt[3]{\frac{5}{9}} + \frac{4}{5}\sqrt[3]{3}$.

It will often be convenient to multiply both terms of a fraction by such a quantity as will make the denominator a perfect power of the degree indicated, in which case, the factor remaining under the sign will be entire. Thus, in the 5th example,

$$\sqrt[3]{\frac{24}{25}} = \sqrt[3]{\frac{8\times3\times5}{125}} = \frac{2}{5}\sqrt[3]{15}\text{; also, } \sqrt[3]{\frac{5}{72}} = \sqrt[3]{\frac{15}{216}} = \frac{1}{6}\sqrt[3]{15}.$$

6. $2\sqrt{\frac{4}{3}} + \sqrt{\frac{3}{5}}$. *Ans.* $\frac{2}{3}\sqrt{15} + \frac{1}{5}\sqrt{15}$.

7. $7\sqrt[3]{\frac{8}{3}} - 4\sqrt[4]{\frac{1}{4}}$. *Ans.* $\frac{14}{3}\sqrt[3]{9} - 2\sqrt[4]{44}$.

8. $\sqrt{150} + \sqrt{1805} + \sqrt[3]{320}$.

Ans. $5\sqrt{6} + 19\sqrt{5} + 4\sqrt[3]{5}$.

9. $\sqrt[4]{768} + \sqrt[n]{a^{m+n}b^n} - \sqrt[5]{608}$.

Ans. $4\sqrt[4]{3} + ab\sqrt[n]{a^m} - 2\sqrt[5]{19}$.

10. $\sqrt{ax^2 - 6ax + 9a}$. *Ans.* $(x-3)\sqrt{a}$.

11. $(a^{4n}b^{2p}c^3)^{\frac{1}{2}} + (a^3 - a^2b)^{\frac{1}{2}}$.

Ans. $a^{2n}b^pc(c)^{\frac{1}{2}} + a(a-b)^{\frac{1}{2}}$.

12. $(a^4b^3c)^{\frac{1}{2}} + (a^5b^3)^{\frac{1}{3}}$. *Ans.* $a^2b(bc)^{\frac{1}{2}} + ab(a^2)^{\frac{1}{3}}$.

13. $\frac{1}{2}\sqrt{\frac{1}{2}} + \frac{1}{3}\sqrt[3]{\frac{1}{3}} + \frac{1}{4}\sqrt[4]{\frac{1}{4}}$. *Ans.* $\frac{1}{4}\sqrt{2} + \frac{1}{9}\sqrt[3]{9} + \frac{1}{8}\sqrt[4]{4}$.

14. $\sqrt{\frac{a^2b + a^2x}{b^3 - b^2x}}$. *Ans.* $\frac{a}{b}\sqrt{\frac{b+x}{b-x}}$.

15. $\left\{\frac{(x+y)^m(x-y)^{2m}}{(a+b)^{4m}}\right\}^{\frac{1}{2m}}$

Ans. $\frac{x-y}{(a+b)^2}\left\{(x+y)^m\right\}^{\frac{1}{2m}} = \frac{x-y}{(a+b)^2}(x+y)^{\frac{1}{2}}$.

To introduce a factor under the radical sign.

128. Take the example, $4a\sqrt{2c}$. Since $\sqrt{16a^2} = 4a$, the given expression may be written,

$$4a\sqrt{2c} = \sqrt{16a^2} \times \sqrt{2c},$$

which, by Principle (1), gives,

$$4a\sqrt{2c} = \sqrt{16a^2 \times 2c} = \sqrt{32a^2c}.$$

In like manner, any factor without the radical sign may be introduced, as a factor, under the radical sign. Hence, the

RULE.

Raise the factor to the power indicated, and write it as a factor under the radical sign.

EXAMPLES.

Transform the following radicals by introducing the coefficients, as factors, under the radical signs:

1. $7\sqrt{3ax}$. *Ans.* $\sqrt{147ax}$.

2. $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$. *Ans.* $\sqrt[3]{\frac{12}{343}}$.

3. $7a\sqrt{2x} + \frac{1}{3}\sqrt[3]{12}$. *Ans.* $\sqrt{98a^2x} + \sqrt[3]{\frac{4}{9}}$.

4. $9\sqrt{3} + 3\sqrt[3]{3} + 2\sqrt[4]{8a^3}$.
Ans. $\sqrt{243} + \sqrt[3]{81} + \sqrt[4]{128a^3}$.

5. $4\sqrt{2a} - 21a\sqrt{3a} - ax\sqrt[3]{x^2}$.

Ans. $\sqrt{32a} - \sqrt{1323a^3} - \sqrt[3]{a^3x^5}$.

6. $4\sqrt{\frac{7}{3}} - 5\sqrt{\frac{1}{2}} + 12\sqrt[3]{\frac{1}{4}}$.

Ans. $\sqrt{\frac{112}{3}} - \sqrt{\frac{25}{2}} + \sqrt[3]{432}$.

This transformation is often useful in finding numerical values of radicals. Thus, it is easier to find the cube root of 432 than to find 12 times the cube root of $\frac{1}{4}$, to which it is equivalent.

To change the index of a radical.

129. Take the radical, $\sqrt[3]{3a}$. Since $3a$ is equal to $\sqrt{3a \times 3a}$, or $\sqrt{9a^2}$, we have,

$$\sqrt[3]{3a} = \sqrt[3]{\sqrt{9a^2}}.$$

But, from Principle (3), $\sqrt[3]{\sqrt{9a^2}} = \sqrt[6]{9a^2}$; hence,

$$\sqrt[3]{3a} = \sqrt[6]{9a^2}.$$

Here, the radical having an index 3 has been transformed to an equivalent radical, having an index 6.

In like manner,

$$\sqrt{2ax} = \sqrt[4]{4a^2x^2} = \sqrt[10]{(2ax)^5} = \sqrt[16]{(2ax)^8}, \text{ \&c.}$$

Again, we have from Principle (3),

$$\sqrt[4]{49a^2x^2} = \sqrt{\sqrt{49a^2x^2}}.$$

But, $\sqrt{49a^2x^2} = 7ax$; hence,

$$\sqrt[4]{49a^2x^2} = \sqrt{7ax}.$$

In like manner, we have,

$$\sqrt[6]{125a^3b^3} = \sqrt{\sqrt[3]{125a^3b^3}} = \sqrt{5ab}.$$

Since we may operate in the same way in all similar cases, we have the following

RULE.

The index of any radical may be multiplied by any number, provided we raise the quantity under the radical sign to a power whose exponent is the same number. Also, the index of a radical may be divided by any number, provided we extract that root of the quantity under the radical sign whose index is the same number.

The same principle is illustrated, when the radical is denoted by fractional exponents, as follows:

$(a)^{\frac{1}{3}}$ is evidently equal to $(a)^{\frac{3}{9}}$, since $\frac{1}{3} = \frac{3}{9}$.

Also, $(a)^{1\frac{3}{12}}$ is equal to $(a)^{\frac{1}{4}}$, since $\frac{3}{12}$ is equal to $\frac{1}{4}$; that is,

$$a^{\frac{1}{3}} = a^{\frac{3}{9}}; \text{ and } a^{1\frac{3}{12}} = a^{\frac{1}{4}},$$

or, by writing the equivalent expressions,

$$\sqrt[3]{a} = \sqrt[9]{a^3}, \qquad \sqrt[12]{a^3} = \sqrt[4]{a}:$$

These are the same results as obtained by the rule. Hence, in the case of fractional exponents, *we may multiply or divide both terms of the fractional exponent without changing the value of the radical.*

EXAMPLES.

Verify the following equations:

1. $\sqrt{3} + \sqrt[3]{4} + \sqrt[4]{5} = \sqrt[4]{9} + \sqrt[6]{16} + \sqrt[8]{25}.$

2. $\sqrt[4]{25} - \sqrt[6]{27} - \sqrt[8]{49} = \sqrt{5} - \sqrt{3} - \sqrt[4]{7}.$

3. $3\sqrt[4]{a^2-2ax+x^2} - 2\sqrt[6]{8a^3x^3} = 3\sqrt{a-x} - 2\sqrt{2ax}.$

4. $\sqrt{a-x} - \sqrt{a+x} = \sqrt[4]{a^2-2ax+x^2} - \sqrt[4]{a^2+2ax+x^2}.$

5. $(ax)^{\frac{1}{2}} + (by)^{\frac{1}{3}} = (ax)^{\frac{2}{4}} + (by)^{\frac{3}{9}}.$

6. $(z)^{\frac{2}{4}} - (y)^{\frac{4}{6}} = (z)^{\frac{1}{2}} - (y)^{\frac{2}{3}}.$

To reduce radicals to a common index.

130. Radicals may be reduced to a common index by means of the preceding principle. Let it be required to reduce the radicals, $\sqrt{a}$, $\sqrt[3]{b}$, and $\sqrt[4]{c}$, to equivalent ones having a common index. Here, the least common multiple of the indices is 12; reducing each to the index 12, by the foregoing principle, we have,

$$\sqrt{a} = \sqrt[12]{a^6}, \quad \sqrt[3]{b} = \sqrt[12]{b^4}, \quad \text{and} \quad \sqrt[4]{c} = \sqrt[12]{c^3}.$$

All other cases may be treated in the same way. Hence, the

RULE.

Find the least common multiple of the indices, and reduce each radical to that index.

When the radicals are expressed by fractional exponents, we have simply to reduce these exponents to a common denominator.

EXAMPLES.

1. Reduce 2, $(3)^{\frac{1}{3}}$, $(a)^{\frac{1}{2}}$, and $(b)^{\frac{1}{4}}$, to a common index.

Ans. $(2^{12})^{\frac{1}{12}}$, $(3^4)^{\frac{1}{12}}$, $(a^6)^{\frac{1}{12}}$, and $(b^3)^{\frac{1}{12}}$.

2. Reduce $a^{\frac{1}{2}}$, b^2, $c^{\frac{2}{3}}$, and $d^{\frac{2}{3}}$, to a common index.

Ans. $(a^3)^{\frac{1}{6}}$, $(b^{12})^{\frac{1}{6}}$, $(c^4)^{\frac{1}{6}}$, and $(d^4)^{\frac{1}{6}}$.

3. Reduce $\sqrt{a+x}$, $\sqrt[3]{a-x}$, and $\sqrt[4]{a^2-x^2}$, to a common index.

Ans. $\sqrt[12]{(a+x)^6}$, $\sqrt[12]{(a-x)^4}$, and $\sqrt[12]{(a^2-x^2)^3}$.

4. Reduce $\sqrt{\frac{2}{3}}$, $\sqrt[3]{2}$, and $5\sqrt{3}$, to a common index.

Ans. $2\sqrt[6]{\frac{8}{27}}$, $\sqrt[6]{4}$, and $5\sqrt[6]{27}$.

5. Reduce ax, $(bx)^{\frac{1}{2}}$, $(cx)^{\frac{1}{3}}$, and $(dx)^{\frac{1}{4}}$, to a common index.

Ans. $(a^{12}x^{12})^{\frac{1}{12}}$, $(b^6x^6)^{\frac{1}{12}}$, $(c^4x^4)^{\frac{1}{12}}$, and $(d^3x^3)^{\frac{1}{12}}$.

6. Reduce cx^2, $(dx^3)^{\frac{1}{4}}$, and $(z^4)^{\frac{1}{2}}$, to a common index.

Ans. $(cx^2)^{\frac{4}{4}}$, $(dx^3)^{\frac{1}{4}}$, and $(z^4)^{\frac{2}{4}}$

7. Reduce $\sqrt[3]{7}$, $\sqrt{10}$, and $\sqrt[6]{\frac{1}{49}}$, to a common index. *Ans.* $\sqrt[6]{49}$, $\sqrt[6]{1000}$, and $\sqrt[6]{\frac{1}{49}}$.

8. Reduce $\sqrt{\frac{1}{2}}$, $\sqrt[4]{\frac{8}{49}}$, and $\sqrt[6]{1331}$, to a common index. *Ans.* $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{4}{7}}$, and $\sqrt{11}$.

ADDITION OF RADICALS.

131. Radicals cannot be added, unless they are similar, any more than feet can be added to pounds. Hence, the first step is to see whether the given radicals are similar, which can only be determined by reducing each

to its simplest form. It will happen, in the general case, that the radicals, when reduced to their simplest forms, have coefficients. Any factor written without the radical sign, is to be regarded as a coefficient. Since the coefficient indicates how many times the radical part is to be taken, we have, for adding similar radicals, the following

RULE.

Reduce the radicals to their simplest forms; then, if they are similar, add the coefficients for a new coefficient, and write the sum before the common radical part. If they are dissimilar, the addition can only be indicated.

EXAMPLES.

Find the sums of the following radicals:

1. $\sqrt{18}$, $\sqrt{32}$, $\sqrt{50}$, and $\sqrt{72}$. *Ans.* $18\sqrt{2}$.

2. $2\sqrt{8}$, $3\sqrt{50}$, and $6\sqrt{18}$. *Ans.* $37\sqrt{2}$.

3. $\sqrt{\frac{2}{5}}$, $\sqrt{\frac{1}{15}}$, and $\sqrt{\frac{1}{4\,0}}$. *Ans.* $\frac{43}{10}\sqrt{15}$.

4. $\frac{2}{3}\sqrt[3]{\frac{3}{9}}$, $\frac{1}{6}\sqrt[3]{\frac{1}{36}}$, and $\frac{3}{5}\sqrt[3]{\frac{3}{32}}$. *Ans.* $\frac{2}{5}\sqrt[3]{6}$.

5. $x\sqrt{12a^4x}$, $2a^2\sqrt{27x^3}$, $3a\sqrt{48a^2x^3}$, and $5a^2x\sqrt{3x}$. *Ans.* $25a^2x\sqrt{3x}$.

6. $\sqrt[3]{54a^{n+6}b^3}$, $a\sqrt[3]{16a^{n-3}b^6}$, and $\sqrt[3]{2a^{4n+9}}$. *Ans.* $(3a^2b + 2b^2 + a^{n+3})\sqrt[3]{2a^n}$.

7. $6\sqrt[6]{4a^2}$, $2\sqrt[3]{2a}$, and $\sqrt[9]{8a^3}$. *Ans.* $9\sqrt[3]{2a}$.

8. $2\sqrt{3}$, $\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$, and $2\sqrt{\frac{3}{16}}$. *Ans.* $\frac{31}{2}\sqrt{3}$.

9. $3b\sqrt[3]{2a^5b^2}$, $7\sqrt[3]{2a^5b^5}$, and $8a\sqrt[3]{2a^2b^5}$. *Ans.* $18ab\sqrt[3]{2a^2b^2}$.

10. $\frac{ab}{b-c}$, and $\sqrt{\frac{a^2b^2}{(b-c)^2}} - \frac{a^2b}{b-c}$.

Ans. $\frac{a(b+\sqrt{bc})}{b-c}$.

SUBTRACTION OF RADICALS.

132. As in addition, the radicals must be similar. The rule results from that for addition.

RULE.

Reduce the radicals to their simplest forms; then, if they are similar, subtract the coefficient of the subtrahend from that of the minuend, and write the remainder before the common radical part. If they are dissimilar, the operation can only be indicated.

EXAMPLES.

1. From $\sqrt{320}$, subtract $\sqrt{80}$. *Ans.* $4\sqrt{5}$.

2. From $b\sqrt[3]{27a^6b}$, subtract $\sqrt[3]{216a^6b^4}$.

Ans. $-3a^2b\sqrt[3]{b}$.

3. From $\sqrt{a^3+2a^2b+ab^2}$, subtract $\sqrt{a^3-2a^2b+ab^2}$.

Ans. $2b\sqrt{a}$.

4. From $\frac{2}{3}\sqrt[3]{\frac{3}{5}} + \frac{3}{5}\sqrt[3]{\frac{5}{3^2}}$, subtract $\frac{1}{6}\sqrt[3]{\frac{1}{3^6}}$.

Ans. $\frac{31}{90}\sqrt[3]{6}$.

5. From $\sqrt{289a^2b}$, subtract $\sqrt{144a^2b}$. *Ans.* $5a\sqrt{b}$.

6. From $2\sqrt{8a^3}+5\sqrt{72a^3}$, subtract $7a\sqrt{18a}+\sqrt{50ab^2}$.

Ans. $(13a-5b)\sqrt{2a}$.

7. From $(a-x)\sqrt{a^2-x^2}$, subtract $\sqrt{\frac{a+x}{a-x}}$.

Ans. $\left(a-x-\frac{1}{a-x}\right)\sqrt{a^2-x^2}$.

8. From $\sqrt[3]{81}+\sqrt[3]{192}$, subtract $\sqrt[3]{512}$.

Ans. $7\sqrt[3]{3}-8$.

MULTIPLICATION OF RADICALS.

133. Since two radicals may always be reduced to a common index, we may take $a\sqrt[n]{b}$, and $c\sqrt[n]{d}$, to represent any two radicals whatever. The indicated product is,

$$a\sqrt[n]{b} \times c\sqrt[n]{d}.$$

We may change the order of the factors without changing the value of the product; we may therefore write the product under the form,

$$ac\sqrt[n]{b} \times \sqrt[n]{d}.$$

But, from Principle (1), $\sqrt[n]{b} \times \sqrt[n]{d} = \sqrt[n]{bd}$; hence,

$$a\sqrt[n]{b} \times c\sqrt[n]{d} = ac\sqrt[n]{bd}.$$

Whence the following

RULE.

Reduce the radicals to a common index; then multiply the coefficients together for a new coefficient, and the quantities under the radical signs for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

Multiply the following radicals:

1. $3\sqrt{8} \times 4\sqrt{48}$. *Ans.* $12\sqrt{384} = 96\sqrt{6}$.

2. $\frac{7}{3}\sqrt[3]{\frac{1}{3}} \times \frac{1}{4}\sqrt[3]{\frac{1}{4}}$. *Ans.* $\frac{7}{12}\sqrt[3]{\frac{1}{3}} = \frac{7}{24}\sqrt[3]{2}$.

3. $4\sqrt{12} \times 3\sqrt{2}$. *Ans.* $12\sqrt{24} = 24\sqrt{6}$.

4. $\frac{1}{3}\sqrt[3]{4} \times \frac{3}{4}\sqrt[3]{12}$. *Ans.* $\frac{1}{4}\sqrt[3]{48} = \frac{1}{2}\sqrt[3]{6}$.

5. $5a\sqrt{ax} \times \frac{5}{2}\sqrt{bx}$. *Ans.* $\frac{25}{2}a\sqrt{abx^2} = \frac{25ax}{2}\sqrt{ab}$.

6. $\sqrt{2ab^3} \times \sqrt{8a^3b}$. *Ans.* $\sqrt{16a^4b^4} = 4a^2b^2$.

7. $\sqrt{8} \times \sqrt[3]{5}$.

Ans. $\sqrt[6]{512} \times \sqrt[6]{25} = \sqrt[6]{12800} = 2\sqrt[6]{200}$.

8. $\sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{3}{4}}$.

Ans. $\sqrt[6]{\frac{1}{4}} \times \sqrt[6]{\frac{27}{64}} = \sqrt[6]{\frac{27}{256}} = \frac{1}{2}\sqrt[6]{\frac{27}{4}}$.

9. $\frac{1}{8}\sqrt[3]{\frac{1}{8}} \times \frac{1}{6}\sqrt[3]{\frac{1}{6}}$. *Ans.* $\frac{1}{48}\sqrt[3]{\frac{1}{48}} = \frac{1}{96}\sqrt[3]{\frac{1}{6}}$.

By combining the above rule with that for the multiplication of polynomials, complicated radical expressions may be multiplied together.

10. Multiply $\sqrt[3]{x} + 2\sqrt[6]{x} + 4$, by $\sqrt[3]{x} + 2\sqrt[6]{x}$.

FIRST OPERATION.

$$\sqrt[3]{x} + 2\sqrt[6]{x} + 4$$
$$\sqrt[3]{x} + 2\sqrt[6]{x}$$
$$\sqrt[3]{x^2} + 2\sqrt[6]{x^3} + 4\sqrt[3]{x}$$
$$+ 2\sqrt[6]{x^3} + 4\sqrt[3]{x} + 8\sqrt[6]{x}$$
$$\sqrt[3]{x^2} + 4\sqrt{x} + 8\sqrt[3]{x} + 8\sqrt[6]{x}$$

SECOND OPERATION.

$$x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 4$$
$$x^{\frac{1}{3}} + 2x^{\frac{1}{6}}$$
$$x^{\frac{2}{3}} + 2x^{\frac{3}{6}} + 4x^{\frac{1}{3}}$$
$$+ 2x^{\frac{3}{6}} + 4x^{\frac{2}{6}} + 8x^{\frac{1}{6}}$$
$$x^{\frac{2}{3}} + 4x^{\frac{1}{2}} + 8x^{\frac{1}{3}} + 8x^{\frac{1}{6}}$$

The two results are identical, but the second has been obtained by following the ordinary rule for exponents. Hence, we conclude that the rule for multiplication is the same whether the exponents are entire or fractional.

11. $(a^{\frac{3}{4}} + a^{\frac{1}{2}}x^{\frac{1}{2}} + a^{\frac{1}{4}}x + x^{\frac{3}{4}}) \times (a^{\frac{1}{4}} - x^{\frac{1}{2}})$.

Ans. $a - x^2$.

12. $\left(x + \frac{p}{2} + \sqrt{q + \frac{p^2}{4}}\right) \times \left(x + \frac{p}{2} - \sqrt{q + \frac{p^2}{4}}\right)$.

Ans. $x^2 + px - q$.

13. $\left(\frac{x}{b}\sqrt{\frac{a}{b}} + \sqrt{\frac{c}{d}}\right) \times \left(\frac{x}{b}\sqrt{\frac{a}{b}} - \sqrt{\frac{c}{d}}\right)$.

Ans. $\frac{ax^2}{b^3} - \frac{c}{d}$.

14. $\left(\sqrt[3]{a^{-\frac{1}{2}}} + \sqrt[6]{a^{\frac{1}{2}}b}\right) \times \left(\sqrt[3]{a^{-\frac{1}{2}}} - \sqrt[6]{a^{\frac{1}{2}}b}\right)$.

Ans. $\sqrt[3]{a^{-1}} - \sqrt[3]{a^{\frac{1}{2}}b}$.

15. $\left(\frac{c}{2} + \frac{1}{2}\sqrt{a^2 - c^2}\right) \times \left(\frac{c}{2} + \frac{1}{2}\sqrt{a^2 - c^2}\right)$.

Ans. $\frac{a^2}{4} + \frac{c}{2}\sqrt{a^2 - c^2}$.

16. $\left(8x^{\frac{3}{4}} + 2x^{\frac{1}{2}}y + \frac{1}{2}x^{\frac{1}{4}}y^2 + \frac{1}{8}y^3\right) \times \left(2x^{\frac{1}{4}} - \frac{y}{2}\right)$.

Ans. $16x - \frac{y^4}{16}$.

DIVISION OF RADICALS.

134. Let $a\sqrt[n]{b}$ and $c\sqrt[n]{d}$ represent any two radicals, after having been reduced to a common index. The quotient of the first by the second may be represented as follows:

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}} = \frac{a}{c} \times \frac{\sqrt[n]{b}}{\sqrt[n]{d}}.$$

But, from Principle (2), $\frac{\sqrt[n]{b}}{\sqrt[n]{d}} = \sqrt[n]{\frac{b}{d}}$; hence,

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}} = \frac{a}{c}\sqrt[n]{\frac{b}{d}}.$$

Whence, the

RULE.

Reduce the radicals to a common index; then divide the coefficient of the dividend by that of the divisor for a new coefficient, and the quantity under the radical sign in the dividend by that in the divisor for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

Perform the following indicated divisions:

1. $\frac{7}{3}\sqrt[3]{\frac{2}{3}} \div \frac{13}{4}\sqrt[3]{\frac{1}{5}}$. *Ans.* $\frac{28}{39}\sqrt[3]{\frac{10}{3}}$.

2. $\frac{1}{4}\sqrt{\frac{2}{5}} \div \frac{3}{7}\sqrt{\frac{5}{2}}$. *Ans.* $\frac{7}{12}\sqrt{\frac{4}{25}} = \frac{7}{30}$.

3. $\frac{1}{2}\sqrt{2ax} \div \frac{3}{4}\sqrt{2bx}$. *Ans.* $\frac{2}{3}\sqrt{\frac{a}{b}}$.

4. $\frac{1}{2}\sqrt{\frac{2}{3}} \div \frac{1}{3}\sqrt[3]{\frac{1}{3}}$. *Ans.* $\frac{1}{2}\sqrt[6]{\frac{8}{27}} \div \frac{1}{3}\sqrt[6]{\frac{1}{9}} = \frac{3}{2}\sqrt[6]{\frac{8}{3}}$.

5. $2\sqrt{2ax} \div \sqrt[3]{4bx^2}$.

Ans. $2\sqrt[6]{8a^3x^3} \div \sqrt[6]{16b^2x^4} = 2\sqrt[6]{\frac{a^3}{2b^2x}}$.

By combining the above rule with that for the division of polynomials, any complicated radical expression may be divided by another.

6. Divide $x + \sqrt{xy} + y$, by $\sqrt{x} + \sqrt[4]{xy} + \sqrt{y}$.

FIRST OPERATION.

$$\begin{array}{l|l}
x + \sqrt{xy} + y & \sqrt{x} + \sqrt[4]{xy} + \sqrt{y} \\
x + \sqrt[4]{x^3y} + \sqrt{xy} & \sqrt{x} - \sqrt[4]{xy} + \sqrt{y} \\
\hline
\quad - \sqrt[4]{x^3y} + y & \\
\quad - \sqrt[4]{x^3y} - \sqrt{xy} - \sqrt[4]{xy^3} & \\
\hline
\qquad\qquad \sqrt{xy} + \sqrt[4]{xy^3} + y & \\
\qquad\qquad \sqrt{xy} + \sqrt[4]{xy^3} + y & \\
\hline
\qquad\qquad 0 &
\end{array}$$

SECOND OPERATION.

$$\begin{array}{l|l}
x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y & x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\
x + x^{\frac{3}{4}}y^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}} & x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\
\hline
\quad - x^{\frac{3}{4}}y^{\frac{1}{4}} + y & \\
\quad - x^{\frac{3}{4}}y^{\frac{1}{4}} - x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}} & \\
\hline
\qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} + y & \\
\qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} + y & \\
\hline
\qquad\qquad 0 &
\end{array}$$

The two results are identical; but the second one has been obtained by following the ordinary rule for exponents. Hence, we conclude that the operation for division is the same whether the exponents are entire or fractional.

7. $\left(16x - \frac{y^4}{16}\right) \div \left(2x^{\frac{1}{4}} - \frac{y}{2}\right)$.

Ans. $8x^{\frac{3}{4}} + 2x^{\frac{1}{2}}y + \frac{1}{2}x^{\frac{1}{4}}y^2 + \frac{1}{8}y^3$.

REDUCTION OF RADICALS.

135. It is often useful to transform radical expressions of the form $\frac{a}{\sqrt{b} + \sqrt{c}}$, and $\frac{a}{\sqrt{b} - \sqrt{c}}$, into equivalent expressions, in which the denominator is rational, that is, when it contains no radical expressions.

The first form may be thus transformed, by multiplying both terms by $\sqrt{b} - \sqrt{c}$; and the second, by multiplying both terms by $\sqrt{b} + \sqrt{c}$, giving

$$\frac{a\sqrt{b} - a\sqrt{c}}{b - c}, \text{ and } \frac{a\sqrt{b} + a\sqrt{c}}{b - c}.$$

If only one term of the denominator is affected by a radical, the same rule will hold good.

EXAMPLES.

Render the denominators of the following fractions rational:

1. $\frac{2}{\sqrt{3} + \sqrt{4}}$. *Ans.* $\frac{2\sqrt{3} - 2\sqrt{4}}{3 - 4}$.

2. $\frac{3}{\sqrt{2} - \sqrt{3}}$. *Ans.* $\frac{3\sqrt{2} + 3\sqrt{3}}{2 - 3}$.

3. $\dfrac{4}{11-2\sqrt{3}}$. *Ans.* $\dfrac{44+8\sqrt{3}}{109} = \dfrac{44}{109} + \dfrac{8}{109}\sqrt{3}$.

4. $\dfrac{3}{8+\sqrt{2}}$. *Ans.* $\dfrac{24-3\sqrt{2}}{62} = \dfrac{12}{31} - \dfrac{3}{62}\sqrt{2}$.

5. $(\sqrt{3}-\sqrt{2}) \div (\sqrt{2}+1)$.

Ans. $\sqrt{2}-\sqrt{3}+\sqrt{6}-2$.

6. $4 \div (\sqrt{5}+1)$. *Ans.* $\sqrt{5}-1$.

7. $(\sqrt{a+x}+\sqrt{a-x}) \div (\sqrt{a+x}-\sqrt{a-x})$.

Ans. $\dfrac{a}{x}+\sqrt{\dfrac{a^2}{x^2}-1}$.

136. In the solution of certain equations, it oftentimes becomes necessary to extract the square root of expressions of the form, $a+\sqrt{b}$, and $a-\sqrt{b}$. In some cases, this operation may be performed, in other cases it cannot be performed. To investigate a rule for determining when the operation can be performed, and the manner of performing it, assume

$$\sqrt{a+\sqrt{b}} = x+y \quad \ldots\ldots (1.)$$
$$\sqrt{a-\sqrt{b}} = x-y \quad \ldots\ldots (2.)$$

Squaring both members of (1) and (2),

$$a+\sqrt{b} = x^2+2xy+y^2 \quad \ldots (3.)$$
$$a-\sqrt{b} = x^2-2xy+y^2 \quad \ldots (4.)$$

Adding (3) and (4), and omitting the common factor 2,

$$a = x^2+y^2 \quad \ldots\ldots (5.)$$

Multiplying (1) by (2),

$$\sqrt{a^2 - b} = x^2 - y^2 \quad \ldots\ldots \quad (6.)$$

Adding and subtracting (5) and (6),

$$x^2 = \frac{a + \sqrt{a^2 - b}}{2} \quad \ldots \quad (7.)$$

$$y^2 = \frac{a - \sqrt{a^2 - b}}{2} \quad \ldots \quad (8.)$$

Now, if $a^2 - b$ is a perfect square, its root may be represented by c. Substituting in (7) and (8), and extracting the square root of each member of both equations (Axiom 5), we have,

$$x = \sqrt{\frac{a + c}{2}}, \text{ and } y = \sqrt{\frac{a - c}{2}}.$$

These values, substituted in (1) and (2), give,

$$\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + c}{2}} + \sqrt{\frac{a - c}{2}} \quad \ldots \quad (9.)$$

$$\sqrt{a - \sqrt{b}} = \sqrt{\frac{a + c}{2}} - \sqrt{\frac{a - c}{2}} \quad \ldots \quad (10.)$$

The square root of the given quantities may be extracted when $a^2 - b$ is a perfect square, and the roots may be obtained by substitution in (9) and (10).

EXAMPLES.

1. Required the square root of $14 + 6\sqrt{5} = 14 + \sqrt{180}$.

Here, $a = 14$, $b = 180$, and $c = \sqrt{196 - 180} = 4$: hence,

$$\sqrt{14 + 6\sqrt{5}} = \sqrt{\frac{14 + 4}{2}} + \sqrt{\frac{14 - 4}{2}} = 3 + \sqrt{5}.$$

2. Required the square root of $18 - 2\sqrt{77}$.

Here, $a=18$, $b=308$, and $c=\sqrt{324-308}=4$; hence,

$$\sqrt{18-2\sqrt{77}} = \sqrt{\frac{22}{2}} - \sqrt{\frac{14}{2}} = \sqrt{11} - \sqrt{7}.$$

3. Required the square root of $94 + 42\sqrt{5}$.

Ans. $7 + 3\sqrt{5}$.

4. Required the square root of $28 + 10\sqrt{3}$.

Ans. $5 + \sqrt{3}$.

OPERATIONS ON IMAGINARY QUANTITIES.

137. An IMAGINARY QUANTITY is an indicated even root of a negative quantity. Thus, $\sqrt{-4}$, $\sqrt[4]{-16}$, $\sqrt{-a^2}$, are imaginary quantities.

The rule deduced for multiplying radicals requires some modification, when applied to imaginary quantities. By the rule already deduced, the product of $\sqrt{-4}$ by $\sqrt{-3}$ would be equal to $\sqrt{12}$; whereas, the true product is $-\sqrt{12}$, as will be shown hereafter.

Every imaginary quantity of the second degree can, by Principle (1), (Art. 125), be resolved into two factors, one of which is $\sqrt{-1}$; the other may be either rational or irrational. Thus,

$$\sqrt{-4} = 2\sqrt{-1},\ \sqrt{-3} = \sqrt{3}\times\sqrt{-1},\ \sqrt{-a^2} = a\sqrt{-1}.$$

The factor, $\sqrt{-1}$, is called the *imaginary factor*, and the other one is called its *coefficient*. Thus, in the expression, $\sqrt{3}\times\sqrt{-1}$, the factor $\sqrt{3}$ is the coefficient of the imaginary factor $\sqrt{-1}$.

When several imaginary factors are to be multiplied

together, we first reduce them to the form, $a\sqrt{-1}$. We can then multiply together the coefficients of the imaginary factor by known rules. It remains to deduce a rule for multiplying the imaginary factor into itself, or what is the same, for raising the imaginary factor to a power whose exponent is equal to the number of factors.

The first power of $\sqrt{-1}$, is $\sqrt{-1}$; the second power, by the definition of square root, is -1; the third power, is the product of the first and second powers, or $-1\sqrt{-1} = -\sqrt{-1}$; the fourth power, is the square of the second, or $+1$; the fifth, is the product of the first and fourth, that is, it is the same as the first; the sixth, is the same as the second; the seventh, the same as the third; the eighth, the same as the fourth; the ninth, again, the same as the first; and so on indefinitely, as shown in the table, n, being any whole number.

$$(\sqrt{-1})^1 = \sqrt{-1}. \qquad (\sqrt{-1})^{4n+1} = \sqrt{-1}.$$
$$(\sqrt{-1})^2 = -1. \qquad (\sqrt{-1})^{4n+2} = -1.$$
$$(\sqrt{-1})^3 = -\sqrt{-1}. \qquad (\sqrt{-1})^{4n+3} = -\sqrt{-1}.$$
$$(\sqrt{-1})^4 = +1. \qquad (\sqrt{-1})^{4n} = 1.$$

To show the use of this table, let it be required to find the continued product of $\sqrt{-4}$, $\sqrt{-3}$, $\sqrt{-2}$, $\sqrt{-7}$, and $\sqrt{-8}$. Reducing these expressions to the proper form, and indicating the multiplication, we have,

$$2\sqrt{-1} \times \sqrt{3}\sqrt{-1} \times \sqrt{2}\sqrt{-1} \times \sqrt{7}\sqrt{-1} \times 2\sqrt{2}\sqrt{-1}.$$

Changing the order of the factors,

$$(2 \times \sqrt{3} \times \sqrt{2} \times \sqrt{7} \times 2\sqrt{2})\,(\sqrt{-1})^5.$$

Hence, the product is equal to, $8\sqrt{21} \times \sqrt{-1} = 8\sqrt{-21}$.

EXAMPLES.

Perform the multiplications indicated below:

1. $\sqrt{-a^2} \times \sqrt{-b^2}$. *Ans.* $a \times b(\sqrt{-1})^2 = -ab$.

2. $\sqrt{-a^2} \times \sqrt{-b^2} \times \sqrt{-c^2}$.
Ans. $abc(\sqrt{-1})^3, = -abc\sqrt{-1}$.

3. $\sqrt{-a^2} \times \sqrt{-b^2} \times \sqrt{-c^2} \times \sqrt{-d^2}$.
Ans. $abcd(\sqrt{-1})^4 = abcd$.

4. $(4 + \sqrt{-2}) \times (3 - \sqrt{-2})$. *Ans.* $14 - \sqrt{-2}$.

5. $(2 - \sqrt{-2}) \times (2 - \sqrt{-2})$. *Ans.* $2 - 4\sqrt{-2}$.

6. $(3 - \sqrt{-2}) \times (3 + \sqrt{-2})$. *Ans.* 11.

From what precedes, it follows that the only radical parts of any power of an expression of the form, $a \pm b\sqrt{-1}$, will be of the form $c\sqrt{-1}$.

Properties of Imaginary Quantities.

138. 1. *A quantity of the form,* $a\sqrt{-1}$, *cannot be equal to the sum of a rational quantity and a quantity of the form,* $b\sqrt{-1}$.

For, if so, let us have the equality,

$$a\sqrt{-1} = x + b\sqrt{-1}.$$

Squaring both members, we have,

$$-a^2 = x^2 + 2bx\sqrt{-1} - b^2.$$

Transposing, and dividing by $2bx$,

$$\sqrt{-1} = \frac{b^2 - a^2 - x^2}{2bx}:$$

an equation which is manifestly absurd, for the first member is imaginary, and the second real, and no imaginary quantity can be equal to a real quantity. Hence, the hypothesis is absurd; and, consequently, the principle enunciated is true.

In the same way, it may be shown that no radical of the second degree can be equal to an entire quantity plus a radical of the second degree.

2. *If,* $a + b\sqrt{-1} = x + y\sqrt{-1}$, *then* $a = x$, *and* $b = y$.

For, by transposition, we have,

$$b\sqrt{-1} = (x - a) + y\sqrt{-1}.$$

But from the preceding principle, this equation can only be true when $x - a = 0$, or $x = a$; making this supposition, and dividing both members of the given equation by $\sqrt{-1}$, we have $b = y$: which was to be shown.

In the same way, it may be shown that, when $a + \sqrt{b} = x + \sqrt{y}$, we shall have, $a = x$, and $b = y$: that is, in all equations of this form, the *rational and radical parts, in each member, are respectively equal to each other.*

3. *The product of two factors, of the forms,* $x - (a + b\sqrt{-1})$, *and* $x - (a - b\sqrt{-1})$, *is positive for all real values of* x.

For, performing the multiplication, we find the product equal to,

$$x^2 - 2ax + a^2 + b^2;$$

which can be written under the form,

$$(x - a)^2 + b^2.$$

Now, whatever may be the value of x, the part $(x-a)^2$ will be positive, since it is a square; b^2 is also positive; hence, their sum, or the required product, is also positive: which was to be proved.

RADICAL EQUATIONS.

139. RADICAL EQUATIONS are equations containing radical quantities. No fixed rules can be given for solving such equations. The methods of proceeding will be best illustrated by examples.

EXAMPLES.

1. Given $\sqrt{x+16} = 2 + \sqrt{x}$, to find x.

Squaring,

$$x + 16 = 4 + 4\sqrt{x} + x.$$

Transposing and reducing,

$$\sqrt{x} = 3.$$

Squaring,

$$x = 9.$$

2. Given $1 - \sqrt{1-x} = n(1 + \sqrt{1-x})$, to find x.

Reducing,

$$1 - n = (n+1)\sqrt{1-x}.$$

Squaring,

$$1 - 2n + n^2 = (n^2 + 2n + 1)(1 - x).$$

Whence,

$$x = \frac{4n}{(1+n)^2}.$$

3. Given $\sqrt{a+x} - \sqrt{a-x} = \sqrt{ax}$, to find x.

Squaring and reducing,

$$2a - ax = 2\sqrt{a^2 - x^2}.$$

Squaring,

$$4a^2 - 4a^2x + a^2x^2 = 4a^2 - 4x^2.$$

Reducing, and dividing by x,

$$(a^2 + 4)x = 4a^2.$$

Whence,

$$x = \frac{4a^2}{a^2 + 4}.$$

4. Given $\dfrac{x-1}{\sqrt{x}+1} = 4 + \dfrac{\sqrt{x}-1}{2}$.

Reducing,

$$\sqrt{x} - 1 = 4 + \frac{\sqrt{x}-1}{2};$$

or,

$$\sqrt{x} - 1 = 8.$$

Transposing,

$$\sqrt{x} = 9; \quad x = 81.$$

5. Given $\dfrac{x}{\sqrt{a^2+x^2}} = \dfrac{c-x}{\sqrt{b^2+(c-x)^2}}$, to find x

Squaring,

$$\frac{x^2}{a^2+x^2} = \frac{(c-x)^2}{b^2+(c-x)^2}.$$

Clearing of fractions,

$$b^2x^2 + x^2(c-x)^2 = a^2(c-x)^2 + x^2(c-x)^2.$$

Reducing,

$$b^2x^2 = a^2(c-x)^2.$$

Extracting square root,

$$bx = a(c-x).$$

$$\therefore \quad x = \frac{ac}{a+b}.$$

6. Given $\dfrac{x-a}{\sqrt{x}+\sqrt{a}} - \dfrac{\sqrt{x}-\sqrt{a}}{3} + 2\sqrt{a}$, to find x

Reducing,

$$\sqrt{x} - \sqrt{a} = \frac{\sqrt{x} - \sqrt{a}}{3} + 2\sqrt{a}.$$

Whence,

$$\sqrt{x} - \sqrt{a} = 3\sqrt{a};$$

or,

$$\sqrt{x} = 4\sqrt{a}.$$

$$\therefore \quad x = 16a.$$

7. Given $8\sqrt{3x} + \frac{81(3 + 4\sqrt{3x})}{16x - 3} = 16x + 3.$

Transposing,

$$16x - 8\sqrt{3}\sqrt{x} + 3 = \frac{81(3 + 4\sqrt{3}\sqrt{x})}{(4\sqrt{x} - \sqrt{3})(4\sqrt{x} + \sqrt{3})}.$$

Factoring,

$$(4\sqrt{x} - \sqrt{3})^2 = \frac{81\sqrt{3}(\sqrt{3} + 4\sqrt{x})}{(4\sqrt{x} - \sqrt{3})(4\sqrt{x} + \sqrt{3})}.$$

Reducing,

$$(4\sqrt{x} - \sqrt{3})^3 = 81\sqrt{3} = 27\sqrt{27}.$$

Extracting cube root,

$$4\sqrt{x} - \sqrt{3} = 3\sqrt{3}.$$

Transposing,

$$4\sqrt{x} = 3\sqrt{3} + \sqrt{3}.$$

Dividing by 4, and squaring,

$$x = \frac{(3\sqrt{3} + \sqrt{3})^2}{16} = 3.$$

Solve the following equations:

8. $\sqrt{x} = 1 + \sqrt{x - 9}.$ *Ans.* $x = 25.$

9. $\sqrt{x} + \sqrt{x - 3} = 3.$ *Ans.* $x = 4.$

10. $\sqrt{x} - \sqrt{2} = \sqrt{x-2}$. *Ans.* $x = 2$.

11. $\sqrt[3]{4x+3} = 3$. *Ans.* $x = 6$.

12. $\sqrt{5x+4} = 2 + \sqrt{3x}$. *Ans.* $x = 12$.

13. $2\sqrt{x} - \sqrt{a} = 2\sqrt{x-a}$. *Ans.* $x = \frac{25a}{16}$.

14. $a + x = \sqrt{x^2 + 5x - a}$. *Ans.* $x = \frac{a^2+a}{5-2a}$.

15. $\sqrt{a-x} = \frac{a}{\sqrt{a-x}} - x$. *Ans.* $x = a - 1$.

16. $\frac{\sqrt{x}-2}{3} + 3 = \frac{x-4}{\sqrt{x}+2}$. *Ans.* $x = 42\frac{1}{4}$.

17. $x - \sqrt{a^2 + x\sqrt{x^2-1}} = a$. *Ans.* $x = \frac{4a^2+1}{4a}$.

18. $\sqrt{x+a} = \sqrt{a} + \sqrt{x-a}$. *Ans.* $x = \frac{5a}{4}$.

19. $\sqrt{x} - \sqrt{a-x} = \frac{\sqrt{x}+\sqrt{a-x}}{2}$. *Ans.* $x = \frac{9a}{10}$.

20. $\sqrt[3]{1+x} + \sqrt[3]{1-x} = \sqrt[3]{2}$. *Ans.* $x = 1$.

21. $\frac{ax-1}{\sqrt{ax}+1} = 4 + \frac{\sqrt{ax}-1}{2}$. *Ans.* $x = \frac{81}{a}$.

22. $\frac{1}{x} + \frac{1}{a} = \sqrt{\frac{1}{a^2} + \sqrt{\frac{4}{b^2x^2} + \frac{1}{x^4}}}$.

Ans. $x = \frac{ab^2}{a^2-b^2}$.

23. $a + x + \sqrt{a + bx + x^2} = b$. *Ans.* $x = \frac{b^2-2ab}{3b-2a}$.

24. $\frac{x-9}{\sqrt{x}+3} + \frac{x-4}{\sqrt{x}-2} = \frac{4(x-16)}{\sqrt{x}+4}$. *Ans.* $x = 56\frac{1}{4}$.

CHAPTER IX.

EQUATIONS OF THE SECOND DEGREE.

EQUATIONS CONTAINING BUT ONE UNKNOWN QUANTITY.

140. EQUATIONS of the second degree, containing but one unknown quantity, may always be reduced to the form of

$$x^2 + 2px = q.$$

Take the equation,

$$\frac{x^2}{2} - \frac{6x}{4} + 5 = 6 - \frac{4x^2}{6} + \frac{(2x+1)^2}{3}.$$

Clearing of fractions by multiplying by 12, and performing indicated operations,

$$6x^2 - 18x + 60 = 72 - 8x^2 + 16x^2 + 16x + 4.$$

Transposing unknown terms to the first member, and known terms to the second member,

$$6x^2 + 8x^2 - 16x^2 - 18x - 16x = 72 + 4 - 60.$$

Factoring and reducing,

$$-2x^2 - 34x = 16.$$

Dividing by the coefficient of x^2, that is, by -2,

$$x^2 + 17x = -8:$$

Which is of the required form; $2p$, in this case, being equal to 17, and q being equal to -8.

All other equations of the same kind may be treated in the same way; hence, the following rule for reducing equations of the second degree, containing but one unknown quantity, to the form of

$$x^2 + 2px = q:$$

RULE.

I. *Clear the equation of fractions, and perform all the indicated operations.*

II. *Transpose all the unknown terms to the first member, and all the known terms to the second member.*

III. *Reduce all the terms containing the square of the unknown quantity to a single term, one factor of which is the square of the unknown quantity; reduce, also, all the terms containing the first power of the unknown quantity to a single term.*

IV. *Divide both members of the resulting equation by the coefficient of the square of the unknown quantity.*

The resulting equation, is called the *reduced equation.*

141. The solution of the reduced equation, consists in finding such values of the unknown quantity as will satisfy the equation, when substituted for the unknown quantity, that is, will make the two members equal. Every such value is called a ROOT.

142. Two cases may arise: *first*, it may happen that $2p$, or the coefficient of the first power of the unknown quantity, is equal to 0; in this case, the equation is said

to be *incomplete: secondly*, it may happen that the coefficient of the first power of the unknown quantity is not equal to 0; in this case the equation is said to be *complete.*

Incomplete equations, when reduced, contain but two terms: one containing the square of the unknown quantity; the other, a known term.

Complete equations, when reduced, contain three kinds of terms, viz.: a term containing the square of the unknown quantity, a term containing the first power of the unknown quantity, and a known term.

First Case. Incomplete Equations.

143. In this case, the reduced equation takes the form,

$$x^2 = q.$$

Extracting the square root of both members, we have,

$$x = \pm\sqrt{q}.$$

Hence, the rule for solving incomplete equations:

RULE.

Reduce the equation to the form, $x^2 = q$, *and extract the square root of both members.*

There will be two roots numerically equal, but having contrary signs. Denoting the first root by x', and the second by x'', we have,

$$x' = +\sqrt{q}, \text{ and } x'' = -\sqrt{q}.$$

EXAMPLES.

Solve the following equations:

1. $x^2 + 5 = \frac{10x^2}{3} - 16.$

Ans. $x' = +3,\ x'' = -3.$

2. $3x^2 - 4 = 28 + x^2.$ *Ans.* $x' = +4,\ x'' = -4.$

3. $\frac{3x^2 + 5}{8} - \frac{x^2 + 29}{3} = 117 - 5x^2.$

Ans. $x' = +5,\ x'' = -5.$

4. $x^2 + ab = 5x^2.$

Ans. $x' = +\frac{1}{4}\sqrt{ab},\ x'' = -\frac{1}{4}\sqrt{ab}.$

5. $(x + a)^2 = 2ax + b.$

Ans. $x' = +\sqrt{b - a^2},\ x'' = -\sqrt{b - a^2}.$

6. $\frac{x + 7}{x^2 - 7x} - \frac{x - 7}{x^2 + 7x} = \frac{7}{x^2 - 73}.$

Ans. $x' = +9,\ x'' = -9.$

7. $x\sqrt{a + x^2} = b + x^2.$

Ans. $x' = +\frac{b}{\sqrt{a - 2b}},\ x'' = \frac{-b}{\sqrt{a - 2b}}.$

8. $\sqrt{\frac{x - 2}{x + 2}} + \sqrt{\frac{x + 2}{x - 2}} = 4.$

Ans. $x' = +\frac{4}{3}\sqrt{3},\ x'' = -\frac{4}{3}\sqrt{3}.$

Second Case. Complete Equations.

144. The reduced form of the complete equation is,

$$x^2 + 2px = q.$$

Adding p^2 to both members (Axiom 1), we have,

$$x^2 + 2px + p^2 = q + p^2.$$

Extracting the square root of both members (Axiom 5), we have,

$$x + p = \pm \sqrt{q + p^2}.$$

Transposing p to the second member, we have,

$$x = -p \pm \sqrt{q + p^2}.$$

Hence, there are two roots, one corresponding to the *plus* sign of the radical, and the other to the *minus* sign. Denoting these roots by x' and x'', we have,

$$x' = -p + \sqrt{q + p^2}, \text{ and } x'' = -p - \sqrt{q + p^2}.$$

Hence, we have the following rule for solving complete equations of the second degree:

RULE.

I. *Reduce the equation to the form,* $x^2 + 2px = q$, *by the rule.*

II. *The first root is equal to half the coefficient of the second term, taken with a contrary sign, plus the square root of the second member increased by the square of half the coefficient of the second term.*

III. *The second root is equal to half the coefficient of the second term, taken with a contrary sign, minus the square root of the second member increased by the square of half the coefficient of the second term.*

1. Let it be required to solve the equation,

$$3x^2 - 14x + 15 = 0.$$

Reducing to the required form,

$$x^2 - \frac{14}{3}x = -5.$$

Writing out the roots,

$$x' = \frac{7}{3} + \sqrt{-5 + \frac{49}{9}}, \text{ and } x'' = \frac{7}{3} - \sqrt{-5 + \frac{49}{9}}$$

Or, reducing,

$$x' = \frac{7}{3} + \frac{2}{3} = 3, \text{ and } x'' = \frac{7}{3} - \frac{2}{3} = \frac{5}{3}.$$

These roots may be verified. Substituting 3 for x, in the given equation, we have,

$$3 \times (3)^2 - 14 \times 3 + 15 = 0; \text{ or, } 0 = 0.$$

Substituting $\frac{5}{3}$ for x, in the same equation, we have,

$$3 \times \left(\frac{5}{3}\right)^2 - 14 \times \frac{5}{3} + 15 = 0; \text{ or, } 0 = 0:$$

which shows that both 3, and $\frac{5}{3}$, are roots.

2. Again, let it be required to solve the equation,

$$4x - \frac{14 - x}{x + 1} = 14.$$

Clearing of fractions, and reducing to the required form,

$$x^2 - \frac{9}{4}x = 7.$$

Writing out the roots, by the rule,

$$x' = \frac{9}{8} + \sqrt{7 + \frac{81}{64}}, \text{ and } x'' = \frac{9}{8} - \sqrt{7 + \frac{81}{64}}.$$

Or, reducing,

$$x' = \frac{9}{8} + \frac{23}{8} = 4, \quad \text{and} \quad x'' = \frac{9}{8} - \frac{23}{8} = -\frac{7}{4};$$

which roots may be verified as before.

3. Given $\frac{3x+4}{5} - \frac{30-2x}{x-6} = \frac{7x-14}{10}$, to find x.

Clearing of fractions, and reducing to the required form, we have,

$$x^2 - 48x = -432.$$

Writing out the two roots,

$$x' = 24 + \sqrt{-432 + 576}, \quad \text{and} \quad x'' = 24 - \sqrt{-432 + 576},$$

Or, reducing,

$$x' = 36, \quad \text{and} \quad x'' = 12.$$

In writing out the roots by the rule, it frequently happens that the quantity under the radical sign is made up of two fractions, or of an entire part and a fraction, as in Examples 1 and 2. In such cases, the two parts must be reduced to the least common denominator which is a perfect square, and then added together, and reduced to the simplest form.

Solve the following equations:

4. $5x^2 - 6x - 60 = 3$. *Ans.* $x' = \frac{21}{5}$, $x'' = -3$.

5. $(x - 12)(x + 2) = 0$.

Ans. $x' = 12$, $x'' = -2$.

6. $ax^2 - bx = c$.

Ans. $x' = \frac{b + \sqrt{b^2 + 4ac}}{2a}$, $x'' = \frac{b - \sqrt{b^2 + 4ac}}{2a}$.

7. $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$. *Ans.* $x' = 3$, $x'' = \frac{21}{11}$.

8. $(x + 2)^2 = 2x^2 + 8$. *Ans.* $x' = 2$, $x'' = 2$.

9. $4x^2 - 9x = 90$. *Ans.* $x' = 6$, $x'' = -\frac{15}{4}$.

10. $\frac{x - 3}{x + 5} - \frac{x + 4}{x - 7} = 2\frac{7}{9}$. *Ans.* $x' = -8\frac{3}{11}$, $x'' = 4$.

11. $x^2 - (a + b)x + ab = 0$. *Ans.* $x' = a$, $x'' = b$.

12. $\frac{4x^2}{3} = \frac{x}{3} + 11$. *Ans.* $x' = 3$, $x'' = -\frac{11}{4}$.

13. $\frac{x}{x + 1} + \frac{x + 1}{x} = 2\frac{1}{6}$. *Ans.* $x' = 2$, $x'' = -3$.

14. $\frac{x + 4}{3} - \frac{4x + 7}{9} = \frac{7 - x}{x - 3} - 1$.

Ans. $x' = 21$, $x'' = 5$

15. $(x - 1)^2 = 2(x^2 + 1)$.

Ans. $x' = -1$, $x'' = -1$

16. $x^2\left(1 - \frac{1}{x}\right) = 8(x + 2)$.

Ans. $x' = \frac{9 + \sqrt{145}}{2}$, $x'' = \frac{9 - \sqrt{145}}{2}$.

17. $17x^2 + 19x - 1848 = 0$.

Ans. $x' = 9\frac{14}{17}$, $x'' = -11$.

18. $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$. *Ans.* $x' = 1\frac{1}{2}$, $x'' = -\frac{1}{6}$.

19. $\frac{2x - 10}{8 - x} - \frac{x + 3}{x - 2} = 2$. *Ans.* $x' = 7$, $x'' = \frac{4}{3}$.

20. $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}$. *Ans.* $x' = 11$, $x'' = -13$.

21. $x + \frac{24}{x-1} = 3x - 4$. *Ans.* $x' = 5$, $x'' = -2$.

22. $\frac{x^2+1}{2x} + \frac{x-1}{4} = 3x - 2$.

Ans. $x' = 1$, $x'' = -\frac{2}{9}$.

23. $x + \frac{1}{x} + 3\frac{(x-1)}{4} = \frac{7}{x}$.

Ans. $x' = \frac{3+\sqrt{681}}{14}$, $x'' = \frac{3-\sqrt{681}}{14}$.

24. $x^3 + (5-x)^3 = 35$. *Ans.* $x' = 3$, $x'' = 2$.

25. $\frac{1200}{x} = \frac{1200}{40+x} + 5$.

Ans. $x' = 80$, $x'' = -120$.

145. Many equations of a higher degree than the second, may be reduced to the form of the second degree, and then solved. One of the most important classes of such equations consists of what are called *Trinomial Equations.* Such equations contain three kinds of terms, viz.: terms involving the unknown quantity to any even degree, terms involving the unknown quantity to a degree half as great, and known terms. Such, for example, as, $x^6 - 4x^3 = 32$, $x^4 - 2x^2 = 3$.

Every trinomial equation may be reduced to the form of,

$$x^{2n} + 2px^n = q,$$

in the same way that equations of the second degree are reduced to the form of,

$$x^2 + 2px = q.$$

When reduced to the required form, we may regard x^n as the unknown quantity, and then they may be solved by the same rule as is given for the solution of equations of the second degree. Having found the two values of x^n, we may find the values of x, by extracting the n^{th} root of these values. To illustrate, let it be required to solve the equation,

26. $x^6 - 4x^3 = 32.$

This is of the required form. Writing out the values of x^3, by the rule, we have,

$$x'^3 = 2+\sqrt{32+4} = 8, \text{ and } x''^3 = 2-\sqrt{32+4} = -4.$$

Whence, by extracting the cube roots of these values, we have,

$$x' = \sqrt[3]{8} = 2, \text{ and } x'' = \sqrt[3]{-4}.$$

27. $x^4 - 2x^2 = 3.$

This is of the required form. Writing out the values of x^2, we have,

$$x'^2 = 1+\sqrt{3+1} = 3, \text{ and } x''^2 = 1-\sqrt{3+1} = -1.$$

Whence, by extracting the square roots of these roots, we have,

$$x' = \pm\sqrt{3}, \text{ and } x'' = \pm\sqrt{-1}.$$

28. $x^4 - 8x^2 = 9.$

Here, $x'^2 = 4+\sqrt{9+16} = 9$, and $x''^2 = 4-\sqrt{9+16} = -1$,

$$\therefore x' = \pm 3. \qquad x'' = \pm\sqrt{-1}.$$

29. $x^3 - x^{\frac{3}{2}} = 56.$

Here,

$$x'^{\frac{3}{2}} = \tfrac{1}{2}+\sqrt{56+\tfrac{1}{4}} = 8, \text{ and } x''^{\frac{3}{2}} = \tfrac{1}{2}-\sqrt{56+\tfrac{1}{4}} = -7$$

Squaring these values,

$$x'^3 = 64, \quad \text{and} \quad x''^3 = 49.$$

Extracting the cube roots of these values, we have,

$$x' = 4, \quad \text{and} \quad x'' = \sqrt[3]{49}.$$

30. $x^6 + 20x^3 = 69$. *Ans.* $x' = \sqrt[3]{3}$, $x'' = \sqrt[3]{-23}$.

Radical equations may, also, by transformation, give rise to equations of the second degree. To illustrate, let it be required to solve the equation,

31. $x + \sqrt{10x + 6} = 9$.

Transposing and squaring,

$$10x + 6 = 81 - 18x + x^2.$$

$$\therefore \quad x^2 - 28x = -75$$

$$x' = 14 + \sqrt{196 - 75} = 25, \quad x'' = 14 - \sqrt{196 - 75} = 3.$$

32. $\dfrac{\sqrt{4x + 20}}{4 + \sqrt{x}} = \dfrac{4 - \sqrt{x}}{\sqrt{x}}$.

Clearing of fractions,

$$\sqrt{4x^2 + 20x} = 16 - x.$$

Squaring, $4x^2 + 20x = 256 - 32x + x^2$.

Whence, $x^2 + \dfrac{52}{3}x = \dfrac{256}{3}$.

$$\therefore \quad x' = -\frac{26}{3} + \sqrt{\frac{256}{3} + \frac{676}{9}} = 4 :$$

$$x'' = -\frac{26}{3} - \sqrt{\frac{256}{3} + \frac{676}{9}} = -\frac{64}{3}.$$

33. $4x + 4\sqrt{x+2} = 7$. *Ans.* $x' = 4\frac{1}{4}$, $x'' = \frac{1}{4}$.

34. $x + \sqrt{5x+10} = 8$. *Ans.* $x' = 18$, $x'' = 3$.

35. $ax + 2\sqrt{n^2x + nax^2} = (3x-1)n$.

Ans. $x' = \frac{n}{n-a}$, $x' = \frac{n}{9n-a}$.

36. $x^4 - 74x^2 = -1225$.

Ans. $x' = \pm 7$, $x'' = \pm 5$.

PROBLEMS.

1. Find two numbers whose difference is 8, and whose product is 128.

Let x denote the lesser number; then will $x+8$ denote the greater number.

From the conditions of the problem,

$$x(x+8) = 128.$$

Solving this equation, we find,

$$x' = 8, \text{ and } x'' = -16; \text{ two solutions.}$$

Hence, the numbers are 8 and 16; also, -16 and -8. Either pair of numbers will satisfy the conditions of the problem, as may be seen by trial.

In the second solution, we find the lesser number, -16, and the greater one, -8. Algebraically speaking, $-16 < -8$, and in general, of two negative quantities, that is algebraically the least which is numerically the greatest.

2. A person travelled 105 miles at a uniform rate. On his return, he travelled 2 miles per hour slower, and was 6 hours longer in making the journey. How many miles did he travel per hour in the first instance?

Let x denote the number of miles per hour.

Then will $\frac{105}{x}$, denote the number of hours required to make the direct journey, and $\frac{105}{x-2}$, the number required to make the return journey.

Since he travelled slower on the return journey, he will take a longer time; and from the conditions of the problem,

$$\frac{105}{x-2} - \frac{105}{x} = 6;$$

whence, by solving the equation, we find,

$$x' = 7, \text{ and } x'' = -5.$$

The first solution is the answer to the given problem, as may be shown by verification. The second solution is the answer to another problem which gives the same equation, and which may be enunciated thus: A person travelled 105 miles, at a uniform rate. On his return, he travelled two miles per hour faster, and made the journey six hours sooner.

Denote the rate of travel on the outward journey, by $-x$: then the rate, on the return journey, will be denoted by $-x+2$; and, since he travelled slower on the outward journey,

$$\frac{105}{-x} - \frac{105}{-x+2} = 6; \text{ or, } \frac{105}{x-2} - \frac{105}{x} = 6;$$

the same equation as before: hence, its solution should give both roots, $x' = 7$, and $x'' = -5$.

On account of the greater generality of algebraic language, both of the above problems are expressed by the same equation. Hence, the solution of the equation ought to give the answers to both.

When, therefore, two answers are found to a problem, both of which will not satisfy the conditions of the problem, that one only must be taken which does satisfy them. The other one belongs to a similar problem, in which certain of the given quantities are to be considered in a contrary sense.

3. *A* and *B* set out from two points at the same time, and travel towards each other. On meeting, it appears that *A* has travelled 30 miles more than *B*, and that *A* could travel *B's* distance in $4\frac{1}{6}$ days, and that *B* could travel *A's* distance in 6 days. How far apart were they when they started?

Let x denote the number of miles that *B* travelled; then will $x + 30$ denote the number that *A* travelled. Since *A* can travel *B's* distance in $4\frac{1}{6}$ days, $\frac{x}{4\frac{1}{6}}$, or $\frac{6x}{25}$, will denote the number of miles that *A* travels in 1 day. In like manner, $\frac{x+30}{6}$, denotes the number of miles that *B* travels in 1 day. The whole distance that *A* travels, divided by the distance he travels in one day, or $\frac{x+30}{\left(\frac{6x}{25}\right)}$, gives the number of days that he travels. The whole distance that *B* travels, divided by the distance he travels in one day, or $\left(\frac{x}{\frac{x+30}{6}}\right)$, gives

the number of days that B travels. But they travel the same number of days each. Hence,

$$\frac{x+30}{\left(\frac{6x}{25}\right)} = \frac{x}{\left(\frac{x+30}{6}\right)}.$$

Clearing of complex fractions,

$$\frac{(x+30)(x+30)}{6} = \frac{6x^2}{25}.$$

Whence, by solution, we find,

$$x' = 150, \text{ and } x'' = -\frac{150}{11}.$$

The first result only satisfies the conditions of the problem. Hence, B travels 150 miles, A, 180 miles, and both together, 330 miles, which is the answer to the question proposed.

4. A, B, and C, can together perform a piece of work, in a certain time. A alone can perform the same work in 6 hours more, B alone in 15 hours more, and C alone in twice the time. In how many hours can they all perform it, working together?

Let x denote the number of hours for all to perform it; then will

$x+6$ denote the number of hours for A to perform it;
$x+15$ " " " B "
and, $2x$ " " " C "

In 1 hour, A can perform a portion of the work denoted by $\frac{1}{x+6}$:

In 1 hour, B can perform a portion of the work denoted by $\frac{1}{x+15}$:

In 1 hour, C can perform a portion of the work denoted by $\frac{1}{2x}$:

In 1 hour, all can perform a portion of the work denoted by $\frac{1}{x}$.

Since the sum of the parts which each can perform, is equal to the part they can all perform, we have,

$$\frac{1}{x+6}+\frac{1}{x+15}+\frac{1}{2x}=\frac{1}{x}.$$

Multiplying both members by $2x(x+6)(x+15)$, and reducing,

$$x^2+7x=30.$$

Whence, by solving,

$$x'=3, \text{ and } x''=-10.$$

The value of x' alone satisfies the conditions of the problem. Hence, the answer is 3 hours.

5. There are two numbers whose sum is 40, and the sum of their squares is 818. What are the numbers?

STATEMENT.

$$x^2+(40-x)^2=818.$$

$\therefore x'=23$, $x''=17$, and $40-x'=17$, $40-x''=23$.

The numbers are 17 and 23.

6. The difference of two numbers is 9; and their sum, multiplied by the greater, is equal to 266. What are the numbers? *Ans.* 14 and 5.

7. The sum of two numbers is 73, and their product 732. What are the numbers? *Ans.* 61 and 12.

8. The sum of two numbers is a, and their product is b. What are the numbers?

Ans. $\frac{a}{2}+\sqrt{-b+\frac{a^2}{4}}$, and $\frac{a}{2}-\sqrt{-b+\frac{a^2}{4}}$.

9. A person travels 48 miles at a uniform rate; a second person travels the same distance two hours sooner, and travels 2 miles per hour more than the first. At what rate did the first travel? *Ans.* 6 miles.

10. A and B start at the same time to travel 150 miles. A travels 3 miles an hour faster than B, and finishes his journey $8\frac{1}{2}$ hours before him. How many miles does A travel per hour. *Ans.* 9 miles.

11. A hollow cubical box, whose sides are three inches thick, requires for its construction $27\frac{1}{8}$ cubic feet of material. How many cubic feet of water will it contain? *Ans.* 64 cubic feet.

12. Two square courts are paved with stones a foot square; the larger court is 12 feet longer than the smaller one, and the number of stones in both pavements is 2120. How long is the smaller pavement? *Ans.* 26 feet.

13. A person distributes 120 dollars amongst a certain number of people. The next day he distributes the same sum amongst a number of people greater by 2. Each of the latter receives 2 dollars less than each of the former. How many were there in each case? *Ans.* 10 and 12.

14. A and B set out to meet each other, being

320 miles apart. *A* travelled 8 miles a day more than *B*, and the number of days before they met, was equal to half the number of miles that *B* travelled in 1 day. How many miles did each travel per day?

Ans. *A*, 24 miles; *B*, 16 miles.

15. A passenger and freight train set out at the same time, the former from New York, and the latter from Albany, distant from each other 144 miles. The passenger train arrived in Albany two hours after they met, and the freight train arrived in New York 8 hours after they met. At what rate did each run?

Ans. 24 and 12 miles.

16. A man, working 10 hours, assisted by a boy 6 hours, does a certain piece of work. If the boy works 10 hours, and the man 6 hours, they will perform $\frac{2}{3}$ of the work. How long will it take them both to do the work, supposing the man to work 5 hours longer than the boy?

Ans. $10\frac{1}{10}$, and $5\frac{1}{10}$ hours.

17. Two partners, *A* and *B*, gained 360 dollars. *A's* money was in trade 12 months, and he received, for principal and profit, 520 dollars. *B's* money was 600 dollars, and was in trade 16 months. How much capital had *A*?

Ans. 400 dollars.

18. *A* and *B* travel, at the same rate, towards New York. At the 50th mile-stone from New York, *A overtakes* a flock of geese, travelling $1\frac{1}{2}$ miles an hour, and 2 hours afterwards meets a stage-coach, travelling $2\frac{1}{4}$ miles per hour. *B overtakes* the geese at the 45th mile-stone, and *meets* the coach 40 minutes before reaching the 31st mile-stone. What is the distance between *A* and *B*?

SOLUTION.

Let x denote the rate of A and B's travel, and suppose the circumstances of the problem to commence when A is at the 50th mile-stone.

When B overtakes the geese, he will have travelled $3\frac{1}{3}x$ miles, 5 of which coincide with 5 of A's miles. Hence, the distance between A and B, is $\frac{10x}{3} - 5$ miles.

A meets the coach at the $(50 - 2x)$ mile-stone, and B meets it at the $\left(31 + \frac{2x}{3}\right)$ mile-stone; the coach travels, meantime, $\left(\frac{8}{3}x - 19\right)$ miles. Hence, dividing by the rate of the coach, we have, $\frac{32x - 228}{27}$, the number of hours between the meetings. Adding to $\frac{8}{3}x - 19$, the distance that A travels in that time, we have, $\left(\frac{8x - 57}{3} + \frac{32x^2 - 228x}{27}\right)$ miles, for the distance between A and B. Hence, from the conditions of the problem,

$$\frac{8x - 57}{3} + \frac{32x^2 - 228x}{27} = \frac{10x - 15}{3}$$

$\therefore x = 9$, and $\frac{10x - 15}{3} = 25$, the number of miles required.

GENERAL PROPERTIES OF EQUATIONS OF THE SECOND DEGREE.

146. It has been shown that every equation of the second degree can be reduced to the form of,

$$x^2 + 2px = q \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Transposing q to the first member, then adding p^2 to,

and subtracting it from, the first member, which may be done without changing its value, we have,

$$(x^2 + 2px + p^2) - (q + p^2) = 0 \quad . \quad . \quad (2.)$$

Factoring the first member, on the principle that the difference of the squares of two quantities is equal to the sum of the quantities multiplied by their difference, we have,

$$(x + p + \sqrt{q + p^2}) \times (x + p - \sqrt{q + p^2}) = 0 \quad (3.)$$

Equation (3) may be satisfied in two ways, and only in two ways, viz.: we may make the second factor of the first member equal to 0, and we may make the first factor equal to 0.

Placing the first factor equal to 0, we have,

$$x + p - \sqrt{q + p^2} = 0. \quad \therefore x = -p + \sqrt{q + p^2} \quad (4.)$$

Placing the second factor equal to 0, we have,

$$x + p + \sqrt{q + p^2} = 0. \quad \therefore x = -p - \sqrt{q + p^2} \quad (5.)$$

These two suppositions give the two roots already found, and those only. Hence, we deduce this principle:

1. *Every equation of the second degree has two roots, and only two.*

If we examine Equation (3), we see that its first member is composed of two factors of the first degree, with respect to x, the first terms of each factor being the unknown quantity, and the second terms being the two roots, each taken with the contrary sign.

Hence, we deduce the following principle:

2. *If all the terms be transposed to the first member, that member may be resolved into two factors of the*

first degree, with respect to the unknown quantity; the first term of each factor being the unknown quantity, and the second term being the two roots, each taken with its sign changed.

The reverse of this principle, enables us to form an equation, when its roots are given, for which operation we may write the following

RULE.

Subtract each root from the unknown quantity, and multiply the results together, placing the product equal to 0.

EXAMPLES.

1. Required the equation whose roots are 2 and -3.

By the rule, $(x-2)\cdot(x+3) = 0$, whence,

$$x^2 + x - 6 = 0; \text{ or, } x^2 + x = 6.$$

2. Required the equation whose roots are 5 and $-\frac{2}{3}$.

$$Ans.\ x^2 - \frac{13}{3}x = \frac{10}{3}.$$

3. Required the equation whose roots are a and b.

$$Ans.\ x^2 - (a+b)x = -ab.$$

4. Required the equation whose roots are $\frac{3}{7}$ and $\frac{7}{3}$.

$$Ans.\ x^2 - \frac{58}{21}x = -1.$$

5. Required the equation whose roots are -7 and -3.

$$Ans.\ x^2 + 10x = -21.$$

6. Required the equation whose roots are $-\frac{3}{4}$ and $\frac{5}{2}$.

$$Ans.\ x^2 - \frac{7}{4}x = \frac{15}{8}.$$

Let us resume the consideration of Equation (1), and its roots:

Adding the two roots together, we have,

$$(-p+\sqrt{q+p^2})+(-p-\sqrt{q+p^2}=-2p.$$

Multiplying the two roots together, we have,

$$(-p+\sqrt{q+p^2})\times(-p-\sqrt{q+p^2})=-q.$$

From these results, we deduce the following additional principles:

3. *The algebraic sum of the two roots is equal to the coefficient of the second term, with its sign changed.*

4. *The product of the two roots is equal to the second member, with its sign changed.*

It follows, from the last principle, that the product of the two roots will be positive when the second member is negative. But the product can only be positive when their signs are alike.

In this case, let us denote the difference of the two roots by $2d$, their sum being $-2p$. Then will the roots be equal to,

$$-p+d, \quad \text{and} \quad -p-d;$$

and their product will be equal to,

$$p^2-d^2.$$

This product will be the greatest possible, when $d=0$, that is, when the roots are equal. In this case, the product is p^2; in all other cases, the product is less than p^2. Hence, we deduce the following additional principle:

5. *When the second member is negative, it can never be numerically greater than the square of half the coefficient of the second term.*

These principles are employed in discussing the general equation of the second degree.

GENERAL DISCUSSION OF EQUATIONS OF THE SECOND DEGREE.

147. We have seen that *the discussion of an equation consists in making every possible supposition on the arbitrary quantities which enter it, and interpreting the results.*

In discussing the equation,

$$x^2 + 2px = q,$$

we shall first make every possible hypothesis with respect to the signs of p and q. It is plain, *first*, that both p and q may be positive; *second*, that p may be negative, and q positive; *third*, that p may be positive, and q negative; and *fourth*, that p and q may be negative. These hypotheses give,

THE FOUR FORMS.

$$x^2 + 2px = q \quad . \ . \ . \ . \quad (1.)$$
$$x^2 - 2px = q \quad . \ . \ . \ . \quad (2.)$$
$$x^2 + 2px = -q \quad . \ . \ . \quad (3.)$$
$$x^2 - 2px = -q \quad . \ . \ . \quad (4.)$$

First Form.

Since q is positive, the product of the roots must be negative (Principle 4); hence, *the roots have contrary signs;* again, since $2p$ is positive, the algebraic sum of the roots is negative (Principle 3); hence, *the negative root is numerically the greater.*

Second Form.

For the same reason as before, *the roots have contrary signs;* since $2p$ is negative, the algebraic sum

of the roots is positive (Principle 3); hence, *the positive root is numerically the greater.*

Third Form

Here, q is negative, and consequently the product of the roots is positive (Principle 4); hence, *the roots have the same sign;* and, since $2p$ is positive, the sum of the roots must be negative (Principle 3); hence, *both roots are negative.*

Fourth Form.

For the same reason as before, *the roots have the same sign;* because $2p$ is negative, the sum of the roots is positive (Principle 3); hence, *the roots are both positive.*

If now, we suppose $p = 0$, and q not 0, the first and second forms reduce to

$$x^2 = q.$$

In this case, the roots have contrary signs, and their sum is equal to 0. Hence, *they are both equal, with contrary signs.*

Under the same supposition, the third and fourth forms become,

$$x^2 = -q.$$

In this case, the roots have the same sign, and their sum is 0, which is impossible. This case comes under the 5th principle.

Solving the equation, we find the roots equal and imaginary. It will always be found, when the second member is negative, and numerically greater than the square of half the coefficient of the second term, that the roots are imaginary, as may be shown by solving either the third or fourth form, and making $p^2 < q$.

Imaginary roots always indicate some absurdity in the conditions that have given rise to the equation whence they are derived, and are to be so interpreted.

If we suppose $q = 0$, and p not equal to 0, the first and third forms reduce to

$$x^2 + 2px = 0,$$

and the second and fourth forms reduce to

$$x^2 - 2px = 0.$$

In both cases, the product of the roots is equal to 0, which shows that one of them must be 0. In the first case, the second root is equal to $-2p$, and in the second, it is equal to $+2p$.

If we suppose, $p = 0$, and $q = 0$, all of the forms reduce to

$$x^2 = 0.$$

In this case, the product of the roots is 0, hence, one of them must be 0, and their sum is equal to 0; hence, the second root is also equal to 0.

The results of the above discussion might be arrived at, by solving the four forms, and discussing the roots thus obtained. Were we to solve them, we should arrive at no results not indicated above.

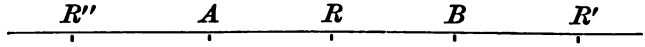
DISCUSSION OF THE PROBLEM OF THE LIGHTS.

148. The problem of the lights affords a fair example of problems of the second degree, and its discussion indicates the general method of reasoning in such cases.

It is a principle of Physics, that the intensity of a light, at any distance, is equal to its intensity at the distance 1, divided by the square of that distance.

PROBLEM.

Having given the intensities of two lights, at the distance 1, and the distance between the lights, it is required to find that point on the line joining them which is equally illuminated by them.



Let A and B represent the places of the two lights, and AB the line joining them. Assume A, as the origin of distances, and call all distances estimated to the right positive, then will all distances to the left be negative.

Denote the distance between the lights by c, the intensity of the light A, at the distance 1, by a, and that of the light B, at the distance 1, by b. Suppose the point R, to be equally illuminated, and denote its distance from the origin A, by x; then will the distance BR, be equal to $c - x$.

Since the intensity of the light A, at the distance 1, is a, at the distance x, it will be $\frac{a}{x^2}$, in accordance with the physical principle enunciated; and since the intensity of the light B, at the distance 1, is b, at the distance $c - x$, it will be equal to $\frac{b}{(c-x)^2}$. From the conditions of the problem, these two expressions must be equal; hence,

$$\frac{a}{x^2} = \frac{b}{(c-x)^2}; \text{ or, } \frac{(c-x)^2}{x^2} = \frac{b}{a}.$$

Extracting the square root of both members, we have,

$$\frac{c-x}{x} = \pm\frac{\sqrt{b}}{\sqrt{a}}.$$

Taking the upper sign, we find,

$$x = \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) \quad . \quad . \quad (1.)$$

Taking the lower sign, we find,

$$x = \frac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right) \quad . \quad . \quad (2.)$$

From the nature of the problem, both a and b are positive, and the two values of x are both real. Hence, there are two points of equal illumination, and only two, on the line of the lights.

To discuss these values of x, we observe that c, a, and b, are, from the nature of the case, arbitrary. They are also positive; and since the conditions of the problem involve the necessity of *two lights*, neither a, b, nor c, can be 0. Hence, only three different suppositions, belonging to the problem, can be made on these arbitrary quantities, viz.:

1. $c > 0$, and $a > b$.
2. $c > 0$, and $a = b$.
3. $c > c$, and $a < b$.

1. $c > 0$, and $a > b$.

In this case, the fraction, $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, is a proper fraction, and therefore less than 1; and because the denominator is less than twice the numerator, it is greater than $\frac{1}{2}$; hence, the first value of x is less than c, and greater than $\frac{1}{2}c$: which shows that the first point is between the two lights, and nearer the lesser one.

The fraction, $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$, is improper, and therefore

greater than 1; hence, the value of x is greater than c: which shows that the second point of equal illumination, is in the prolongation of AB, and on the side of the lesser light.

These results correspond with the supposition. For, if A is the greater light, it is plain that the first point of equal illumination ought to be nearer the second light, and a little reflection will show that there ought to be a second point of equal illumination beyond the second light.

2. $c > 0$ and $a = b$.

In this case, the fraction, $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, is equal to $\frac{1}{2}$, and the fraction, $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$, is equal to $\frac{\sqrt{a}}{0}$, or, ∞. Hence, the first value of x is equal to $\frac{1}{2}c$, and the second is *infinite:* which shows that the first point of equal illumination is midway between the two lights, and the second is at an infinite distance, that is, there is no second point at any finite distance.

These results correspond with the supposition. For, if the lights are of equal intensity, the first point ought to be midway between them, and it is equally plain that there should be no other point on the line of the lights.

3. $c > 0$, and $a < b$.

In this case, the fraction, $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, is less than $\frac{1}{2}$, because the denominator is more than twice the numerator. The fraction, $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$, is negative, because,

$\sqrt{a} < \sqrt{b}$. Hence, the first value of x is positive, and less than $\frac{1}{2}c$, and the second value of x is negative: which shows that the first point is between the lights, and nearer the lesser light, and that the second point is on the prolongation of the lights to the left of A, that is, on the side of the feebler light.

These results correspond with the supposition, the case being just the same as the first, the position of the lights being reversed. The coincidence of the mathematical results with those of ordinary reasoning, shows that the methods of interpretation of positive and negative symbols are correct.

$$c = 0, \quad a > b, \quad a = b, \quad \text{and} \quad a < b.$$

The hypothesis of $c = 0$, places the two lights at the same point, in which case they form one and the same light, whose intensity is equal to the sum of their intensities taken separately.

The conditions of the problem involve the necessity of *two lights*, and the equation of the problem is found under this hypothesis. This equation ought not, therefore, to respond to the case of a single light. For, the interpretation of the results obtained from an equation, can only give the cases which fall under the hypothesis. The hypothesis of two lights, and the hypothesis of a single light, are not connected by any law which affords a common equation of condition. Hence, the results obtained on the supposition of $c = 0$, do not belong to the problem.

EQUATIONS INVOLVING MORE THAN ONE UNKNOWN QUANTITY.

149. In the general case, two equations involving two unknown quantities, which are both of the second degree, cannot be solved by any of the preceding methods. Every attempt to solve two such equations, leads to an equation of the fourth degree, which as yet we have not learned how to solve. When, however, one of the equations is of the first degree, they can always be solved, and there are many particular cases in which they can be solved when both are of the second degree.

Two equations containing two unknown quantities, one of which is of the first and the other of the second degree, can always be solved.

1. Take the two equations,

$$x^2 + 12xy + y^2 = 85 \ldots\ldots (1.)$$

$$x + 3y = 11 \ldots\ldots (2.)$$

Find the value of x in terms of y from (2), and substituting it in (1), we have,

$$121 - 66y + 9y^2 + 132y - 36y^2 + y^2 = 85;$$

or, by reduction, $y^2 - \frac{33}{13}y = \frac{18}{13}$:

whence, $y' = 3$, and $y'' = -1\frac{6}{13}$,

which, substituted in (2), gives,

$$x' = 2, \text{ and } x'' = 12\frac{5}{13}.$$

In like manner, other similar groups of equations may be solved.

Solve the following groups of simultaneous equations:

2. $\left.\begin{aligned} x^2 + y^2 &= 202 \\ x + y &= 20 \end{aligned}\right\}$ *Ans.* $\begin{cases} x' = 11, & x'' = 9. \\ y' = 9, & y'' = 11. \end{cases}$

3. $\left.\begin{aligned} x^2 + y^2 &= 394 \\ x - y &= 2 \end{aligned}\right\}$ *Ans.* $\begin{cases} x' = 15, & x'' = -13. \\ y' = 13, & y'' = -15. \end{cases}$

4. $\left.\begin{aligned} x^2 - 2xy + y^2 &= 9 \\ x + y &= 11 \end{aligned}\right\}$ *Ans.* $\begin{cases} x' = 7, & x'' = 4. \\ y' = 4, & y'' = 7. \end{cases}$

5. $\left.\begin{aligned} x + y &= 6 \\ x^2 + y^2 &= 26 \end{aligned}\right\}$ *Ans.* $\begin{cases} x' = 5, & x'' = 1. \\ y' = 1, & y'' = 5. \end{cases}$

6. $\left.\begin{aligned} x^2 - y^2 &= 16 \\ x + y &= 8 \end{aligned}\right\}$ *Ans.* $\begin{cases} x' = 5. \\ y' = 3. \end{cases}$

Two equations of the second degree, can always be solved when they are homogeneous with respect to the unknown quantities.

To illustrate, let us take the equations:

7. $x^2 + xy = 10$ . . . . . (1.)
$y^2 + xy = 15$ . . . . . (2.)

Assume $y = nx$, n being an auxiliary unknown quantity. Substituting in (1) and (2),

$$x^2 + nx^2 = 10. \qquad \therefore\ x^2 = \frac{10}{n+1} \quad \cdot\ \cdot\ (3.)$$

$$n^2x^2 + nx^2 = 15. \qquad \therefore\ x^2 = \frac{15}{n(n+1)} \quad \cdot\ \cdot\ (4.)$$

Equating these values of x^2, we find,

$$\frac{10}{n+1} = \frac{15}{n(n+1)}; \qquad \therefore\ n = \frac{3}{2}.$$

Substituting this value of n in (3), and the resulting value of x in (1), we have,

$$x = 2, \text{ and } y = 3.$$

Only a single pair of values of x and y are deduced. The complete solution would give four pairs of values.

In the same way, similar groups of equations may be solved.

8. $\left.\begin{aligned} x^2 + y^2 &= 61 \\ x^2 - xy &= 6 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= 6. \\ y &= 5. \end{aligned}\right.$

9. $\left.\begin{aligned} x^2 + xy + y^2 &= 37 \\ x^2 - xy + y^2 &= 13 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= 3. \\ y &= 4. \end{aligned}\right.$

10. $\left.\begin{aligned} x^2 - 2xy &= 5 \\ y^2 + x^2 &= 29 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= 5. \\ y &= 2. \end{aligned}\right.$

11. $\left.\begin{aligned} 3x^2 &= 2xy + 24 \\ y^2 &= xy - 3 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= 4. \\ y &= 3. \end{aligned}\right.$

12. $\left.\begin{aligned} 4xy - 3y^2 &= 64 \\ 2xy + 2x^2 - y^2 &= 138 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= 7. \\ y &= 4. \end{aligned}\right.$

Many other equations of the second degree may be so transformed as to come within the rules for solution. Equations of a higher degree than the second may, also, in certain cases, be reduced, by transformation, to such forms as to come within the rules already given. No general rules can be laid down for making the transformations, each case, in general, requiring to be treated in a manner peculiar to itself. A few examples are given, to illustrate some of the methods of solution.

13. $\left.\begin{aligned} x^2 + y^2 + x + y &= 922 \quad \ldots\ldots\ (1.) \\ \sqrt{xy} &= 20 \quad \ldots\ldots\ (2.) \end{aligned}\right\}$

Squaring (2), and multiplying by 2,

$$2xy = 800 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Adding (1) and (3),

$$(x^2 + 2xy + y^2) + (x + y) = 1722.$$

Regarding $(x + y)$ as a single quantity, we have, by the rule,

$$x + y = -\tfrac{1}{2} \pm \sqrt{1722 + \tfrac{1}{4}} = -\tfrac{1}{2} \pm \tfrac{83}{2}.$$

$$\therefore\ x + y = 41, \text{ and } x + y = -42.$$

Taking the first value of $x + y$, and combining with (3), we have,

$$x = 25, \quad y = 16.$$

14. $$\left.\begin{aligned} x + y + \sqrt{x + y} &= 12 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.) \\ x^3 + y^3 &= 189 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.) \end{aligned}\right\}$$

From (1) taking only the first value, we have,

$$\sqrt{x + y} = -\tfrac{1}{2} + \sqrt{12 + \tfrac{1}{4}} = -\tfrac{1}{2} + \tfrac{7}{2} = 3 :$$

$$\therefore\ x + y = 9 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Dividing (2) by (3), member by member,

$$x^2 - xy + y^2 = 21 \quad . \quad . \quad . \quad . \quad (4.)$$

Squaring (3),

$$x^2 + 2xy + y^2 = 81 \quad . \quad . \quad . \quad . \quad (5.)$$

Subtracting (4) from (5),

$$3xy = 60; \text{ or, } xy = 20 \quad . \quad . \quad . \quad . \quad (6.)$$

Combining (6) and (3),

$$x = 5, \text{ and } y = 4.$$

15. $$\left.\begin{aligned} x - y &= 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.) \\ x^4 + y^4 &= 272 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.) \end{aligned}\right\}$$

Raising (1) to 4th power,

$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 16 \quad . \quad . \quad (3.)$$

Subtracting (3) from (2), and factoring,

$$xy(4x^2 - 6xy + 4y^2) = 256 \quad . \quad . \quad (4.)$$

Multiplying (1) by 2, squaring, multiplying by xy, and subtracting from (4),

$$2x^2y^2 = 256 - 16xy;$$

or,
$$x^2y^2 + 8xy = 128.$$

$$\therefore \quad xy = -4 \pm \sqrt{128 + 16} = -4 \pm 12;$$

or,
$$xy = 8, \quad xy = -16.$$

Taking the first value $xy = 8$, and combining with (1),

$$x = 4, \quad y = 2.$$

16. $\left.\begin{aligned} x^2 + 2xy + y + 3x &= 73 \\ y^2 + x + 3y &= 44 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 4. \\ y = 5. \end{cases}$

17. $\left.\begin{aligned} xy &= 6 \\ 3x^2 - 7y^2 + 1 &= 0 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 3. \\ y = 2. \end{cases}$

18. $\left.\begin{aligned} x^4 - 2x^2y + y^2 &= 49 \\ x^4 - 2x^2y^2 + y^4 - x^2 + y^2 &= 20 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 3. \\ y = 2. \end{cases}$

19. $\left.\begin{aligned} \frac{x}{y} - \frac{y}{x} &= \frac{11}{30} \\ x^2 + xy &= 66 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 6. \\ y = 5. \end{cases}$

20. $\left.\begin{aligned} x^2y^4 + y^2 &= 10 \\ xy^2 + y &= 4 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 3. \\ y = 1. \end{cases}$

21. $\left.\begin{aligned} x^3 + y^3 &= 189 \\ x^2y + xy^2 &= 180 \end{aligned}\right\}$ *Ans.* $\begin{cases} x = 5. \\ y = 4. \end{cases}$

22. $\left.\begin{aligned}\frac{x+y}{x-y} &= a^2\\ x^2-y^2 &= b^2\end{aligned}\right\}$ *Ans.* $\begin{cases} x=\frac{b}{2a}(a^2+1). \\ y=\frac{b}{2a}(a^2-1). \end{cases}$

23. $\left.\begin{aligned}9x^2 &= 4y^2\\ 3xy+2x+y &= 485\end{aligned}\right\}$ *Ans.* $\begin{cases} x=10. \\ y=15. \end{cases}$

24. $\left.\begin{aligned}x^2+y^2-x-y &= 78\\ xy+x+y &= 39\end{aligned}\right\}$ *Ans.* $\begin{cases} x=9. \\ y=3. \end{cases}$

25. $\left.\begin{aligned}\frac{1}{y}-\frac{1}{x} &= \frac{1}{4}\\ x^2y-xy^2 &= 16\end{aligned}\right\}$ *Ans.* $\begin{cases} x=4. \\ y=2. \end{cases}$

PROBLEMS.

1. Find two such numbers, that their product, added to their sum, shall be 47, and their sum, taken from the sum of their squares, shall leave 62.

Let x and y denote the numbers; then, from the conditions of the problem,

$$\left.\begin{aligned}(x+y)+xy &= 47 \quad \ldots \quad (1.)\\ x^2+y^2-(x+y) &= 62 \quad \ldots \quad (2.)\end{aligned}\right\}$$

Multiplying equation (1) by 2,

$$2xy+2(x+y) = 94 \quad \ldots \quad (3.)$$

Adding (2) and (3),

$$x^2+2xy+y^2+(x+y) = 156.$$

or,

$$(x+y)^2+(x+y) = 156 \quad \ldots \quad (4.)$$

Solving (4), and taking the first value of $x + y$,

$$x + y = -\tfrac{1}{2} + \sqrt{156 + \tfrac{1}{4}} = 12 \quad . \quad . \quad . \quad . \quad (5.)$$

Substituting in (1), we have,

$$xy = 47 - 12 = 35 \quad . \quad . \quad . \quad . \quad (6.)$$

Combining (5) and (6),

$x = 5$, and $y = 7$: the numbers required.

2. The sum of two numbers is 7, and the sum of their cubes is 91. What are the numbers?

Let x and y denote the numbers. Then we shall have,

$$\left.\begin{aligned} x + y &= 7 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.) \\ x^3 + y^3 &= 91 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.) \end{aligned}\right\}$$

Cubing (1), we have,

$$x^3 + 3x^2y + 3xy^2 + y^3 = 343 \quad . \quad . \quad . \quad (3.)$$

Subtracting (2) from (3), factoring, and dividing by 3,

$$xy(x + y) = 84 \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Dividing (4) by (1), member by member,

$$xy = 12 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

Combining (1) and (5), we find,

$x = 3$, and $y = 4$: the numbers required.

3. The fore wheel of a coach makes 6 revolutions more than the hind wheel, in going 120 yards; but if the circumference of each wheel be increased 1 yard, the fore wheel will only make 4 more revolutions than the hind wheel in going the same distance. What is the circumference of each wheel?

Let x denote the number of yards in the circumference of the hind wheel, and y the number in that of the fore wheel.

From the conditions of the problem,

$$\frac{120}{x} + 6 = \frac{120}{y},$$
$$\frac{120}{x+1} + 4 = \frac{120}{y+1}.$$

Or, by reduction,

$$\left.\begin{aligned} 20(x-y) &= xy \quad \ldots\ldots \quad (1.) \\ 29x - 31y &= xy + 1 \quad \ldots\ldots \quad (2.) \end{aligned}\right\}$$

Subtracting (1) from (2),

$$9x - 11y = 1; \text{ or, } x = \frac{11y+1}{9}.$$

Substituting this value of x in (1), and reducing,

$$y^2 - \frac{39}{11}y = \frac{20}{11}.$$

Whence, $\quad y' = 4, \text{ and } y'' = -\frac{5}{11}.$

Taking the positive value of y, and substituting in (1), we find,

$$x = 5.$$

Hence, the wheels are, respectively, 4 and 5 yards in circumference.

4. Find two numbers whose product is equal to the difference of their squares, and the sum of whose squares is equal to the difference of their cubes.

Denoting the numbers by x and y, we have,

$$\left.\begin{aligned} xy &= x^2 - y^2 \quad \ldots\ldots \quad (1.) \\ x^2 + y^2 &= x^3 - y^3 \quad \ldots\ldots \quad (2.) \end{aligned}\right\}$$

Making $x = ny$, n being unknown, we have, from (1), after reduction,

$$n = n^2 - 1 \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Whence, taking the positive value of n, only,

$$n = \frac{1 + \sqrt{5}}{2}; \quad \therefore \quad n^2 = \frac{3 + \sqrt{5}}{2}, \quad \text{and } n^3 = 2 + \sqrt{5}.$$

From (2), by reduction,

$$n^2 + 1 = (n^3 - 1)y; \quad \therefore \quad y = \frac{n^2 + 1}{n^3 - 1} = \frac{5 + \sqrt{5}}{2 + 2\sqrt{5}}.$$

$$x = ny = \frac{1 + \sqrt{5}}{2} \times \frac{5 + \sqrt{5}}{2 + 2\sqrt{5}} = \frac{5 + \sqrt{5}}{4}.$$

5. Required two numbers, whose product is equal to the square of two thirds the first, and the difference of whose squares is greater, by 1, than the square of twice the second. *Ans.* 9 and 4.

6. Find two numbers, whose sum, multiplied by the greater, is 209, and whose difference, multiplied by the less, is 24. *Ans.* 11 and 8.

7. Find two numbers, such that the sum of their squares is equal to 181, and the difference of their squares equal to 19. *Ans.* 9 and 10.

CHAPTER X.

RATIO, PROPORTION, AND SERIES.

RATIO AND PROPORTION.

150. THE MEASUREMENT of a quantity is the operation of finding how many times it contains some other quantity of the same kind, taken as a standard. The latter quantity is called the UNIT OF MEASURE, and may be any quantity which is of the same kind as the quantity measured. The unit of measure is supposed to be known beforehand, and is therefore called the ANTECEDENT. The value of the quantity measured becomes known, in terms of the antecedent, by the operation of measurement, and is therefore called the CONSEQUENT. Mathematically, the measurement is performed by dividing the consequent by the antecedent; the result is an abstract number, which, being prefixed to the unit employed, expresses the value of the thing measured. Hence, the following definition:

151. *The* RATIO *of one quantity to another of the same kind, is the result obtained by dividing the second quantity by the first. The first quantity is called the* ANTECEDENT, *and the second the* CONSEQUENT.

Thus, the ratio of a to b, is $\frac{b}{a}$, or $a)b$; or, it may

be written, $a : b$. Here, a is the antecedent, or standard, and b the consequent, and both are called *terms*.

152. A PROPORTION is *the expression of equality between two equal ratios.* Thus,

$$\frac{b}{a} = \frac{d}{c},$$

is a proportion, expressing the fact that the ratio of a to b is equal to the ratio of c to d. This proportion may also be written thus,

$$a : b :: c : d.$$

In either case, the proportion is read, *a is to b as c is to d.* In most algebraical operations, the former method of writing a proportion is the better; in geometrical operations, the latter method has been most frequently employed.

153. There are four *terms* in every proportion which have received different names with respect to each other. The first and third are *antecedents;* the second and fourth are *consequents.* The first and fourth are *extremes;* the second and third are *means.* The first and second form the *first couplet;* the third and fourth form the *second couplet.* The fourth term, is called a *fourth proportional to the other three.* When the second term is equal to the third, it is said to be a mean proportional between the other two. In this case, there are but three different terms in the proportion, and the last term is said to be a *third proportional to the other two.* Thus, in the proportion,

$$\frac{b}{a} = \frac{d}{c}, \text{ or } a : b :: c : d,$$

a and c are antecedents, b and d consequents, a and d extremes, b and c means; $\frac{b}{a}$, or $a : b$, is the first couplet, $\frac{d}{c}$, or $c : d$, is the second couplet, and d is a fourth proportional to a, b, and c. Also, in the proportion,

$$\frac{b}{a} = \frac{c}{b}, \text{ or } a : b :: b : c,$$

b is a mean proportional between a and c, and c is a third proportional to a and b.

154. Quantities are in proportion, by *alternation*, when antecedent is compared with antecedent, and consequent with consequent.

155. Quantities are in proportion, by *inversion*, when antecedents are made consequents, and consequents are made antecedents.

156. Quantities are in proportion, by *composition*, when the sum of antecedent and consequent is compared with either antecedent or consequent.

157. Quantities are in proportion, by *division*, when the difference of antecedent and consequent is compared with either antecedent or consequent.

158. Two varying quantities are *reciprocally*, or *inversely* proportional, when one is increased as many times as the other is diminished. In this case, their product is a fixed quantity, as $xy = m$.

159. Equimultiples of two quantities, are the results obtained by multiplying both by the same quantity. Thus, ma, and mb, are equimultiples of a and b, whatever may be the value of m.

160. We shall proceed to demonstrate some of the most important properties of proportions, adopting both methods of writing proportions.

Assume the proportion,

$$a : b :: c : d; \text{ or, } \frac{b}{a} = \frac{d}{c} \quad . \quad . \quad . \quad (1.)$$

Clearing of fractions, we have,

$$bc = ad \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Hence,

1. *If four quantities are in proportion, the product of the means is equal to the product of the extremes.*

Conversely, if we divide both members of (2) by ac, we shall have, $\frac{b}{a} = \frac{d}{c}$; or, $a : b :: c : d$. Hence,

If the product of two quantities is equal to the product of two other quantities, the first two may be made the means, and the second two the extremes, of a proportion.

It follows, that of three proportional quantities, *the square of the mean is equal to the product of the extremes.*

If we multiply both members of (1) by $\frac{c}{b}$, and reduce the result to its simplest form, we shal have,

$$\frac{c}{a} = \frac{d}{b}; \text{ or, } a : c :: b : d \quad . \quad . \quad (3.)$$

Whence, the following principle:

2. *If four quantities are in proportion, they will be in proportion by alternation.*

Let us assume the proportion,

$$a : b :: g : f; \quad \text{or, } \frac{b}{a} = \frac{f}{g} \quad . \quad . \quad (4.)$$

Comparing (4) with (1), we see that their first members are equal; hence, their second members must also be equal; that is,

$$\frac{d}{c} = \frac{f}{g}; \text{ or, } c : d :: g : f \quad . \quad . \quad . \quad (5.)$$

Hence, the following principle:

3. *If the first couplets of two proportions are the same, the second couplets will form a proportion.*

Consequently, by alternation, *if the antecedents of two proportions are the same in both, the consequents will be in proportion.*

If we take the reciprocals of both members of (1), we shall have,

$$\frac{a}{b} = \frac{c}{d}; \text{ or, } b : a :: d : c \quad . \quad . \quad . \quad (6.)$$

Hence, the following principle:

4. *If four quantities are in proportion, they will be in proportion by inversion.*

If we add 1 to both members of (1), and also subtract 1 from both members, we shall have,

$$\frac{b}{a} + 1 = \frac{d}{c} + 1, \text{ and } \frac{b}{a} - 1 = \frac{d}{c} - 1.$$

Whence, by reducing to a common denominator, we have,

$$\left.\begin{array}{l} \dfrac{b+a}{a} = \dfrac{d+c}{c}, \qquad \text{and} \qquad \dfrac{b-a}{a} = \dfrac{d-c}{c} \\ a : b+a :: c : d+c, \quad \text{and} \quad a : b-a :: c : d-c \end{array}\right\} (7.)$$

Hence, the following principle:

5. *If four quantities are in proportion, they will be in proportion by composition or division.*

If we multiply both terms of the ratio $\frac{b}{a}$, by m, its value will not be changed (Art. 61), and we shall have,

$$\frac{mb}{ma} = \frac{b}{a}; \text{ or, } ma : mb :: a : b \quad . \quad . \quad (8.)$$

Hence, the following principle:

6. *Equimultiples of two quantities are proportional to the quantities themselves.*

If we multiply both terms of the first member of (1), by m, and both terms of the second member by n, the equality will not be destroyed, and we shall have,

$$\frac{mb}{ma} = \frac{nd}{nc}; \text{ or, } ma : mb :: nc : nd \quad . \quad (9.)$$

Hence, the following principle:

7. *If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.*

If, in Equation (8), we suppose $m = 1 \pm \frac{p}{q}$, in which $\frac{p}{q}$ is any fraction whatever, we shall have the following principle:

8. *If two quantities be increased or diminished by like parts of each, the results will be proportional to the quantities themselves.*

If, in Equation (9), we suppose, $m = 1 \pm \frac{p}{q}$, and $n = 1 \pm \frac{p'}{q'}$, we shall have the following principle:

9. *If both terms of the first couplet of a proportion be increased or diminished by like parts of each, and if both terms of the second couplet be increased or diminished by any other like parts of each, the results will be in proportion.*

A *continued proportion*, is one in which several ratios are successively equal to each other, as,

$$\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \frac{h}{g}, \text{ \&c.; or, } a : b :: c : d :: e : f :: g : h, \text{ \&c. . . . . (10.)}$$

From the preceding continued proportion, we evidently have the following equations:

$$\frac{b}{a} = \frac{b}{a}; \text{ whence, } ba = ab.$$

$$\frac{b}{a} = \frac{d}{c} \quad \text{“} \quad bc = ad.$$

$$\frac{b}{a} = \frac{f}{e} \quad \text{“} \quad be = af.$$

$$\frac{b}{a} = \frac{h}{g} \quad \text{“} \quad bg = ah, \text{ \&c.}$$

Adding and factoring, we have,

$$b(a + c + e + g + \text{\&c.}) = a(b + d + f + h + \text{\&c.})$$

Changing this equation into a proportion (Principle 1), we have,

$$a + c + e + g + \text{\&c.} : b + d + f + h + \text{\&c.} :: a : b; \text{ or,}$$

$$\frac{b + d + f + h + \text{\&c.}}{a + c + e + g + \text{\&c.}} = \frac{b}{a} \quad . \quad . \quad (11.)$$

Hence, the following principle:

10. *In any continued proportion, the sum of all the antecedents is to the sum of all the consequents, as any antecedent is to the corresponding consequent.*

Let us assume the two equations,

$$\frac{b}{a} = \frac{d}{c}, \text{ and } \frac{f}{e} = \frac{h}{g}.$$

Multiplying these equations, member by member, we have,

$$\frac{bf}{ae} = \frac{dh}{cg}; \text{ or, } ae : bf :: cg : dh \quad . \quad (12.)$$

Hence, the following principle:

11. *If two proportions be multiplied together, term by term, the products will be proportional.*

This principle may be extended to the multiplication of any number of proportions, term by term.

SERIES.

161. A SERIES is a succession of terms, each of which is derived from one or more of the preceding ones, by a fixed law. This law is called the *law of the series.*

When a certain number of terms are given, and the law of the series is known, any number of terms may be found. There are, in general, an infinite number of terms in every series.

The simplest series are, Arithmetical and Geometrical Progressions.

ARITHMETICAL PROGRESSION.

162. An ARITHMETICAL PROGRESSION is a series in which each term is derived from the preceding term, by adding to it a constant quantity. This quantity is called the *common difference.*

When the common difference is *positive*, each term is greater than the preceding, and the progression is said to be *increasing.* When the common difference is *negative*, each term is less than the preceding, and the progression is said to be *decreasing.*

Thus, 2, 4, 6, 8, &c., is an *increasing* arithmetical progression, in which the common difference is + 2.

The series, 18, 16, 14, 12, &c., is a *decreasing* arithmetical progression, in which the common difference is − 2.

Although there are an infinite number of terms in every progression, it is customary to speak of any finite number of consecutive ones, as constituting a progression. Thus, we call the succession of terms,

3, 5, 7, 9, 11,

a progression of 5 terms.

If the terms of any increasing progression be taken in a reverse order, beginning at the last, there will result a decreasing progression. Thus, the progression,

4, 8, 12, 16,

becomes, when reversed,

16, 12, 8, 4.

If a decreasing progression, be, in like manner reversed, there will result an increasing progression.

163. In every arithmetical progression, having a finite number of terms, there are five quantities especially to be considered, viz.: *The first term, the last term, the number of terms, the common difference*, and *the sum of the terms*. When any three of these are given, the other two may be found. In investigating rules for the solution of these different cases, let us denote

The first term by . . . . . a.
" last term by . . . . . l.
" number of terms by . . . . n.
" common difference by . . . . d.
" sum of the terms by . . . . s.

The first and last terms are called *extremes*, all the other terms are called *Arithmetical means*.

Given a, d, and n, to find l.

164. The second term is, by definition, equal to $a + d$; the third is equal to the second, increased by d, that is, it is $a + 2d$; the fourth term is equal to the third, increased by d, that is, it is $a + 3d$; and so on: hence, the n^{th} term, or l, is equal to $a + (n-1)d$; or,

$$l = a + (n - 1)d \quad . \quad . \quad . \quad . \quad (1.)$$

That is, *any term is equal to the first term*, plus *the product of the common difference by the number of preceding terms.*

EXAMPLES.

1. The first term is 3, and the common difference is 3. What is the 7th term.

Here, the number of terms preceding the 7th, is 6; hence, by the rule, the 7th term is, $3 + 3 \times 6$, or 21

2. The first term is 24, and the common difference -3; what is the 5th term?

Ans. $24 + 4 \times -3 = 12.$

3. The first term is 1, and the common difference is $1\frac{1}{2}$; what is the 25th term?

Ans. $1 + 24 \times 1\frac{1}{2} = 37.$

4. The first term is 1, and the common difference $-\frac{3}{4}$; what is the 13th term?

Ans. $1 + 12 \times -\frac{3}{4} = -8.$

5. The first term is -5, and the common difference 2; what is the 7th term?

Ans. $-5 + 2 \times 6 = 7.$

Given a, l, and n, to find s.

165. Having given a, d, and n, l may be found by the preceding rule. Having found the n^{th} term, l, the preceding term is equal to $l - d$; the term preceding that, $l - 2d$, and so on. If now, we write out the terms of the progression, and then write the same terms in a reverse order, the sum will be the same in both cases.

Hence, we have,

$$s = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l.$$
$$s = l + (l-d) + (l-2d) + \ldots (a+2d) + (a+d) + a.$$

Adding these equations, term by term, we have,

$$2s = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l).$$

Here, $(a+l)$ is taken n times; hence,

$$2s = n(a+l); \text{ or, } s = \tfrac{1}{2}n(a+l) \quad . \quad . \quad (2.)$$

That is, *the sum of the terms is equal to the sum of the extremes, multiplied by half the number of terms.*

Formula (2) can be placed under another form, by substituting for l its value taken from Formula (1).

$$s = \tfrac{1}{2}n(a+a+(n-1)d) = na + \frac{n(n-1)}{2}d, \quad (3.)$$

by means of which, the sum of the terms may be found more directly than by Formula (2).

EXAMPLES.

1. The first term is 2, the common difference is 3, and the number of terms is 17. What is their sum?

$$Ans.\ 17 \times 2 + \frac{17 \times 16}{2} \times 3 = 442.$$

2. The first term is $\frac{1}{2}$, the common difference $-\frac{1}{8}$, and the number of terms 20. What is the sum?

$$Ans.\ 20 \times \frac{1}{2} + \frac{20 \times 19}{2} \times -\frac{1}{8} = -\frac{55}{4}.$$

3. The first term is 20, the common difference is -2, and the number of terms is 6. What is the sum?

$$Ans.\ 6 \times 20 + \frac{6 \times 5}{2} \times -2 = 90.$$

4. The first term is 5, the common difference 3, and the number of terms 12. What is the sum?

Ans. 258.

5. The first term is -2, the common difference is -3, and the number of terms is 10. What is the sum? *Ans.* -155.

Formulas (1) and (2), contain five quantities: a, d, n, l, and s. If any three be assumed at pleasure, the remaining two may be deduced from the formulas.

GEOMETRICAL PROGRESSION.

166 A GEOMETRICAL PROGRESSION is a series, each term of which is derived from the preceding one, by multiplying it by a fixed quantity. This quantity is called the *ratio* of the progression.

167. When the first term is positive, and the ratio greater than 1, each term is greater than the preceding one, and the progression is said to be *increasing.* When the ratio is less than 1, each term is less than the preceding one, and the progression is said to be *decreasing.*

Thus, the series 2, 4, 8, 16, &c., is an increasing progression, whose ratio is 2.

The series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c., is a decreasing progression, whose ratio is $\frac{1}{2}$.

168. When the ratio is negative, the terms of the progression are alternately positive and negative. The positive terms make up a progression whose ratio is equal to the square of the given ratio, and the negative terms make up a second progression, having the same ratio.

The progression 2, -4, $+8$, -16, &c., whose ratio is -2, is made up of the two progressions, 2, 8, 32, &c., and -4, -16, -64, &c., whose ratio in each case is 4.

169. In any geometrical progression, there are five quantities considered, any three of which being given, the other two may be found These quantities are:

The *first term*, denoted by . . . . . a.
" *last term*, denoted by . . . . . l.
" *number of terms*, denoted by . . n.
" *ratio*, denoted by . . . . . r.
" sum of the terms, denoted by . . s.

The first and last terms are called *extremes;* all the other terms are called *Geometrical means.*

Given a, r, and n, to find l.

170. The second term is, by definition, equal to the first multiplied by r, that is, it is equal to ar; the third term is equal to the second, multiplied by r, that is, it is equal to ar^2; the fourth term is equal to the third, multiplied by r, that is, it is equal to ar^3; and so on to the n^{th} term, which is equal to ar^{n-1}; hence,

$$l = ar^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

That is, *any term of a geometrical progression is equal to the first term, multiplied by that power of the ratio whose exponent is equal to the number of preceding terms.*

EXAMPLES.

1. Find the 7th term of the series 1, 4, 16, &c.

We have, $l = ar^{n-1} = 1 \times 4^6 = 4096$. *Ans.*

2. Find the 8th term of the series 2, 4, 8, &c.
Ans. 256.

3. Find the 12th term of the series 30, 15, $7\frac{1}{2}$, &c.
Ans. $\frac{15}{1024}$.

4. Find the 8th term of the series 5, 25, 125, &c.
Ans. 390625.

Given a, r, and n, to find s.

171. We have, from the definition,

$$s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1};$$
$$\therefore rs = \quad ar + ar^2 + \quad\quad ar^{n-2} + ar^{n-1} + ar^n$$

Subtracting the first of the above equations from the second, member from member, and factoring,

$$s(r-1) = a(r^n - 1);$$
$$\therefore s = a\frac{r^n - 1}{r - 1} \quad \ldots\ldots \quad (2.)$$

Had we subtracted the second from the first, we should have found,

$$s = a\frac{1 - r^n}{1 - r} \quad \ldots\ldots \quad (3.)$$

When r is greater than 1, Formula (2) will be found more convenient; when it is less than 1, Formula (3) is to be preferred; but either may be used in any case.

EXAMPLES.

1. Find the sum of 8 terms of the series 5, 20, 80, &c.

$$s = a\frac{r^n - 1}{r - 1} = 5 \cdot \frac{4^8 - 1}{4 - 1} = 109225. \quad \textit{Ans.}$$

2. Find the sum of 7 terms of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, &c.

Ans. $\frac{2059}{1458}$.

3. Find the sum of 6 terms of the series 64, 32, 16, &c. *Ans.* 126.

4. Find the sum of 8 terms of the series 2, -4, $+8$, -16, &c. *Ans.* -170.

172. Solving Formula (1), with respect to r, we have,

$$r = \sqrt[n-1]{\frac{l}{a}}, \quad \ldots\ldots \quad (4.)$$

a formula, by means of which we can find the *ratio*, when the extremes and the number of terms are given. The same formula enables us to insert any number of geometrical means between any two numbers. Since the number of means is equal to the whole number of terms diminished by 2, we shall have, $n - 1 = m + 1$, in which m denotes the number of means required.

Substituting in (4), we have,

$$r = \sqrt[m+1]{\frac{l}{a}} \quad \ldots\ldots \quad (5.)$$

EXAMPLES.

1. Insert 4 geometrical means between $\frac{1}{3}$ and 81.

We have, $r = \sqrt[5]{81 \div \frac{1}{3}} = \sqrt[5]{243} = 3$. Hence, the means are, 1, 3, 9, and 27.

2. Insert 3 geometrical means between 2 and $\frac{81}{8}$.

Ans. 3, $\frac{9}{2}$, and $\frac{27}{4}$.

3. Insert 4 geometrical means between 2 and 486.

Ans. 6, 18, 54, 162.

4. Insert 5 geometrical means between 1 and $\frac{1}{64}$.

Ans. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$.

5. Insert 4 geometrical means between 2 and 6250.

Ans. 10, 50, 250, 1250.

Formula (3) may be placed under the form,

$$s = \frac{a}{1-r} - \frac{ar^n}{1-r} \quad \ldots \quad (6.)$$

When r is a fraction less than 1, and n very great, the numerator, ar^n, is very small with respect to the denominator; and finally, when $n = \infty$, the value of ar^n is 0. In this case, the value of s reduces to its first term, and we have,

$$s = \frac{a}{1-r} \quad \ldots \quad (7.)$$

173. In all cases of decreasing progression, the value, $\frac{a}{1-r}$, is that towards which the sum of the series approaches, as the number of terms is increased. This term is called, the *limit of the sum.*

EXAMPLES.

1. Find the sum of the terms, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c., to infinity.

We have, $s = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = 4$. *Ans.*

2. Find the sum of the terms, $\frac{4}{3}$, 1, $\frac{3}{5}$, &c., to infinity. *Ans.* $4\frac{1}{6}$.

3. Find the sum of the terms, 1, $\frac{1}{3}$, $\frac{1}{9}$, &c., to infinity. *Ans.* $\frac{3}{2}$.

4. Find the sum of the terms, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{8}$, &c., to infinity. *Ans.* $2\frac{2}{3}$.

5. What is the limit towards which the sum of the series 1, $\frac{4}{5}$, $\frac{16}{25}$, &c., approaches, as the number of terms increases? *Ans.* 5.

6. What is the limit of the sum of the series 1, $\frac{3}{4}$, $\frac{9}{16}$, &c.? *Ans.* 4.

PROBLEMS.

1. The sum of three terms in geometrical progression is 21, and the sum of their reciprocals is $\frac{7}{12}$. What are the terms?

Denote the first term by x, and the ratio by y. We shall have, from the conditions of the problem,

$$x + xy + xy^2 = 21 \quad . \ . \ . \ . \quad (1.)$$
$$\frac{1}{x} + \frac{1}{xy} + \frac{1}{xy^2} = \frac{7}{12} \quad . \ . \ . \ . \quad (2.)$$

From (1) and (2), we find,

$$1 + y + y^2 = \frac{21}{x} \quad . \ . \ . \ . \quad (3.)$$

$$1 + y + y^2 = \frac{7}{12}xy^2 \quad . \ . \ . \quad (4.)$$

$$\therefore \ \frac{21}{x} = \frac{7}{12}xy^2; \text{ or, } x^2y^2 = 36; \text{ or, } xy = 6.$$

Substituting the value of xy in (4),

$$1 + y + y^2 = \frac{7}{12} \times 6y.$$

$$\therefore \ y^2 - \frac{5}{2}y = -1.$$

$$\text{Or, } y = \frac{5}{4} \pm \sqrt{\frac{25}{16} - 1}.$$

Or, using the positive value of y only, we have,

$$y = 2; \text{ whence, } x = 3.$$

The terms are, therefore, 3, 6, and 12.

2. The population of a town increases annually in a geometrical ratio, and in three years the population rises

from 120000 to 138915. Find by what part of itself it is increased?

Let x denote the ratio after each increase.

From the conditions of the problem,

$$120000x^3 = 138915;$$

Or,
$$8000x^3 = 9261.$$

$$\therefore \quad 20x = 21; \text{ or, } x = 1\tfrac{1}{20}.$$

Hence, each year, there is added one-twentieth of its population.

3. The sum of a geometrical progression to an infinite number of terms is 2, and the sum of the squares of the terms of the same series is $\frac{4}{3}$. What is the first term, and the ratio of the given progression?

Let x denote the first term, and y the ratio. We shall have,

$$x + xy + xy^2 + \&c. = \frac{x}{1-y} = 2, \quad (1.)$$

$$x^2 + x^2y^2 + x^2y^4 + \&c. = \frac{x^2}{1-y^2} = \frac{4}{3}. \quad (2.)$$

Dividing (2) by (1),

$$\frac{x}{1+y} = \frac{2}{3}; \text{ or, } 3x = 2 + 2y \quad . \quad . \quad (3.)$$

From (1), we have,

$$x = 2 - 2y \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

From (3) and (4), by combination,

$$x = 1, \text{ and } y = \tfrac{1}{2}.$$

The series is, therefore,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \&c.$$

4. The population of a town increases annually in geometrical progression, and in four years is raised from 10000 to 14641. By what part of itself is it increased each year? *Ans.* $\frac{1}{10}$.

5. Find 4 numbers in geometrical progression, such that the sum of the means shall be 36, and the sum of the extremes shall be 84. *Ans.* 3, 9, 27, and 81.

6. Insert 3 geometrical means between $\frac{1}{2}$ and $3\frac{1}{2}$. *Ans.* $\frac{3}{4}$, $\frac{9}{8}$, $\frac{27}{16}$.

INDETERMINATE COEFFICIENTS.

174. An identical equation is one which is true for all possible values of the unknown quantities which enter it. Thus,

$$ax + b = ax + b, \quad \frac{a^2 - x^2}{a - x} = a + x,$$

are identical equations.

175. Every identical equation containing but one unknown quantity, can be reduced to the form of,

$$p + qx + rx^2 + \&c. = p' + q'x + r'x^2 + \&c. \quad (1.)$$

Or, by transposition, to the form,

$$(p - p') + (q - q')x + (r - r')x^2 + \&c. = 0. \quad (2.)$$

176. From the definition above given, the unknown quantity may have an infinite number of values, that is, it is indeterminate (Art. 74).

177. The coefficients of the different powers of the unknown quantity in Equation (2) being coefficients of

indeterminate quantities, are called INDETERMINATE COEFFICIENTS.

178. We have, from Equation (2), by transposition,

$$(q - q')x + (r - r')x^2 + \&c. = p' - p \quad . \quad . \quad (3.)$$

From the definition of an identical equation (Art. 174), it must be of such a form as to be satisfied for *every possible value* that may be attributed to x. Hence, Equations (1), (2), and (3), must be true for every value of x; consequently, for the value of $x = 0$. Substituting this value in the first member of Equation (3), reduces it to zero: hence, $p' - p = 0$, or $p = p'$.

Making $p = p'$, in Equation (3), and dividing through by x, we have,

$$q - q' + (r - r')x + \&c. = 0.$$

In the same manner as before, it may be shown that $q = q'$, and by continuing the same course of reasoning, we can show that $r = r'$, and so on; that is,

The coefficients of the different powers of the unknown quantity, in Equation (2), *are separately equal to* 0; *or the coefficients of the like powers of the unknown quantity in the two members of Equation* (1) *are separately equal to each other.*

This principle, called the principle of indeterminate coefficients, may be enunciated as follows:

In any identical equation, containing but one indeterminate quantity, the coefficients of the like powers of this quantity in the two members, are separately equal to each other.

179. If an identical equation contains more than one indeterminate quantity, it may be shown by a similar

course of reasoning, that the coefficients of the like powers and combinations of powers of these quantities, in the two members, are separately equal to each other. Hence, the following general principle:

In any identical equation containing any number of indeterminate quantities, the coefficients of the like powers and combinations of powers of these quantities, in the two members, are separately equal to each other.

The principle of indeterminate coefficients is much used in developing expressions into series. To illustrate, let it be required to develop the expression, $\frac{2+3x}{3+4x+5x^2}$, into a series.

Assume the equation,

$$\frac{2+3x}{3+4x+5x^2} = p + qx + rx^2 + sx^3 + \&c. \quad (1.)$$

and suppose it identical.

Clearing of fractions, we have,

$$\begin{array}{l|l|l|l} 2+3x = 3p+4p & x+5p & x^2+5q & x^3+\&c. \\ \quad\quad +3q & \quad +4q & \quad +4r & \\ & \quad +3r & \quad +3s & \end{array}$$

From the principle of indeterminate coefficients, we have,

$$\begin{array}{lll} 2 = 3p, & \therefore & p = \frac{2}{3}. \\ 3 = 4p + 3q, & \therefore & q = \frac{1}{9}. \\ 0 = 5p + 4q + 3r, & \therefore & r = -\frac{34}{27}. \\ 0 = 5q + 4r + 3s, & \therefore & s = \frac{121}{81}, \ \&c. \end{array}$$

Substituting these values in (1), we have the required series,

$$\frac{2+3x}{3+4x+5x^2} = \frac{2}{3} + \frac{1}{9}x - \frac{34}{27}x^2 + \frac{121}{81}x^3, \ \&c.$$

In like manner, any similar expression may be developed into a series. Hence, the following

RULE.

Assume the expression equal to a series of the form, $p + qx + rx^2 +$ &c., *in which* p, q, r, &c., *are quantities to be determined. Clear the equation of fractions, and place the coefficients of the like powers of the unknown quantity in the two members separately equal to each other. Then find; from the resulting equations, the values of* p, q, r, &c., *and substitute these values in the assumed development.*

EXAMPLES.

1. Develop $\frac{1-x}{1+x}$, into a series.

OPERATION.

$$\frac{1-x}{1+x} = p + qx + rx^2 + sx^3 + \text{ &c.}$$

Clearing of fractions,

$$\begin{array}{rl|l|l|l} 1 - x = & p + q & x + r & x^2 + s & x^3 + \text{ &c.} \\ & + p & + q & + r & \end{array}$$

Equating coefficients,

$$\begin{aligned} 1 &= p, & \therefore\ p &= 1. \\ -1 &= q + p, & \therefore\ q &= -2. \\ 0 &= r + q, & \therefore\ r &= 2. \\ 0 &= s + r, & \therefore\ s &= -2, \text{ &c.} \end{aligned}$$

Substituting,

$$\frac{1-x}{1+x} = 1 - 2x + 2x^2 - 2x^3, \text{ &c.}$$

The law of the series is evident, and any number of terms may be written.

2. Develop $\frac{1-x}{1+x+x^2}$, into a series.

OPERATION.

$$\frac{1-x}{1+x+x^2} = p + qx + rx^2 + sx^3 + \&c.$$

Clearing of fractions,

$$\begin{array}{rl|l|l|l} 1 - x = & p + q & x + r & x^2 + s & x^3 + \&c. \\ & + p & + q & + r & \\ & & + p & + q & \end{array}$$

Equating coefficients,

$$\begin{aligned} 1 &= p, & \therefore\ p &= 1. \\ -1 &= q + p, & \therefore\ q &= -2. \\ 0 &= r + q + p, & \therefore\ r &= 1. \\ 0 &= s + r + q, & \therefore\ s &= 1, \ \&c. \end{aligned}$$

Substituting,

$$\frac{1-x}{1+x+x^2} = 1 - 2x + x^2 + x^3 - 2x^4 + x^5 + x^6 - 2x^7, \ \&c.$$

3. Develop $\frac{1+2x}{1-3x}$, into a series.

Ans. $1 + 5x + 15x^2 + 45x^3 + 135x^4 + \&c.$

4. Develop $\frac{1+2x}{1-x-x^2}$, into a series.

Ans. $1 + 3x + 4x^2 + 7x^3 + \&c.$

When the numerator of the fraction contains the unknown quantity to a higher power than the denomi-

nator, we first reduce it to a mixed quantity, in which the numerator of the fractional part is of a less degree than the denominator, and then develop the fractional part, by the rule, annexing the result to the entire part.

5. Develop $\frac{1 + ax^2 - x^3}{a - x}$, into a series.

Reducing to a mixed quantity,

$$\frac{1 + ax^2 - x^3}{a - x} = x^2 + \frac{1}{a - x}.$$

Developing, by the rule,

$$\frac{1}{a - x} = \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \text{\&c.}$$

$$\therefore \frac{1 + ax^2 - x^3}{a - x} = x^2 + \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \frac{x^3}{a^4} + \text{\&c.}$$

180. When the denominator of a fraction can be resolved into factors, the fraction itself can be resolved into partial fractions having these factors for denominators, by the principle of indeterminate coefficients. The preceding remark is also applicable here, namely, when the degree of the numerator is higher than that of the denominator, the fraction must be changed to a mixed quantity, as before. The method of proceeding is indicated in the following example:

1. Resolve the fraction, $\frac{2a^2}{a^2 - x^2}$, into partial fractions.

OPERATION.

Assume, $\frac{2a^2}{a^2 - x^2}, = \frac{p}{a + x} + \frac{q}{a - x}$, in which p and q are to be determined.

Clearing of fractions,

$$2a^2 = pa - px + qa + qx.$$

Equating the coefficients of the like powers of x,

$$\left.\begin{aligned} 2a^2 &= pa + qa \\ 0 &= -p + q \end{aligned}\right\}$$

Whence, $q = a$, and $p = a$.

Substituting in the assumed equation,

$$\frac{2a^2}{a^2 - x^2} = \frac{a}{a - x} + \frac{a}{a + x}.$$

2. Resolve $\frac{x^2 - x + 2}{(x-1)(x-2)(x-3)}$, into partial fractions.

OPERATION.

Assume,

$$\frac{x^2 - x + 2}{(x-1)(x-2)(x-3)} = \frac{p}{x-1} + \frac{q}{x-2} + \frac{r}{x-3}.$$

Clearing of fractions,

$$x^2 - x + 2 = p(x^2 - 5x + 6) + q(x^2 - 4x + 3) + r(x^2 - 3x + 2).$$

Equating coefficients,

$$\left.\begin{aligned} 2 &= 6p + 3q + 2r \\ -1 &= -5p - 4q - 3r \end{aligned}\right\}$$
$$1 = p + q + r.$$

$$\therefore\ p = 1,\ q = -4,\ \text{and}\ r = 4.$$

Substituting in the assumed equation,

$$\frac{x^2 - x + 2}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{4}{x-2} + \frac{4}{x-3}.$$

3. Resolve the fraction, $\frac{3x-5}{x^2-6x+8}$, into partial fractions.

Ans. $\frac{7}{2}\cdot\frac{1}{x-4}-\frac{1}{2}\cdot\frac{1}{x-2}$.

4. Resolve $\frac{5-x}{1-x^2}$, into partial fractions.

Ans. $\frac{2}{1-x}+\frac{3}{1+x}$.

5. Resolve $\frac{3-2x-4x^2}{(1+2x)(1-2x)(1-x)}$, into partial fractions.

Ans. $\frac{1}{1+2x}+\frac{1}{1-2x}+\frac{1}{1-x}$.

When the denominator is of the second degree, it can always be resolved into two factors of the first degree, by placing it equal to 0, and finding the roots of the equation; then, the factors will be found by subtracting each root from the unknown quantity.

We have hitherto supposed that the factors of the denominator are unequal. When the factors are, some of them, equal, we proceed as in the following example:

6. Resolve $\frac{3x^2-7x+6}{(x-1)^3}$, into partial fractions.

Assume, $\frac{3x^2-7x+6}{(x-1)^3}=\frac{p}{(x-1)^3}+\frac{q}{(x-1)^2}+\frac{r}{x-1}$.

Clearing of fractions,

$$3x^2-7x+6=p+q(x-1)+r(x^2-2x+1).$$

Equating coefficients,

$$\left.\begin{aligned} 6 &= p-q+r \\ -7 &= q-2r \\ 3 &= r \end{aligned}\right\}$$

$\therefore\ p=2,\ q=-1,$ and $r=3.$

Substituting in the assumed equation,

$$\frac{3x^2 - 7x + 6}{(x-1)^3} = \frac{2}{(x-1)^3} - \frac{1}{(x-1)^2} + \frac{3}{x-1}.$$

7. Resolve $\frac{x}{(x-1)^2}$, into partial fractions.

Ans. $\frac{1}{(x-1)^2} + \frac{1}{x-1}$.

8. Resolve $\frac{x + 2x^2}{(x-1)^2}$, into partial fractions.

Ans. $\frac{3x}{(x-1)^2} + \frac{2x}{(x-1)}$.

9. Resolve $\frac{1 - x + x^2}{(x-1)^3}$, into partial fractions.

Ans. $\frac{1}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{1}{x-1}$.

10. Resolve $\frac{x^2}{(x^2-1)(x-2)}$, into partial fractions.

Ans. $\frac{4}{3(x-2)} - \frac{1}{2(x-1)} + \frac{1}{6(x+1)}$.

Other applications of the principle of indeterminate coefficients will be given hereafter.

181. Fractions and radical expressions may be developed into series, in many cases, by means of the Binomial Formula. As yet, the Binomial Formula has only been demonstrated for positive and entire exponents, in which case, we have seen that the number of terms in the development will always be finite. When, however, the exponent is negative or fractional, the number of terms will be infinite, that is, the development will be a true series. The method of developing expressions into series, as also the method of summing series, will be explained in the Appendix, after the general demonstration of the Binomial Formula.

CHAPTER XI.

LOGARITHMS.

182. THE LOGARITHM of a number is the exponent of the power to which it is necessary to raise a fixed number to produce the given number. The fixed number is called the *base of the system.*

183. If we denote any positive number, except 1, by a; any positive number whatever by n, and the exponent of the power to which it is necessary to raise a, in order to produce n, by x, we shall have the exponential equation,

$$a^x = n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

In this equation, a is the *base*, n any positive number, and x is the logarithm of n. It is plain, that a cannot be negative, neither can it be equal to 1, because all powers of 1 are equal to 1.

If, whilst a remains fixed in value, we suppose n to assume in succession every value from 0 to ∞, the corresponding values of x, taken together, will constitute what is called *a system of logarithms.* Since there are an infinite number of different values that may be attributed to a, it follows, that *there are an infinite number of systems of logarithms.* Of these, two only are in general use, viz.: that system whose base is 10,

called the *common system;* and that system whose base is 2.718281828..., called the *Naperian system.*

In what follows, we shall designate common logarithms by the symbol *log*, Naperian logarithms by the symbol *l*, and logarithms taken in any system whatever by the symbol *Log.*

184. If we make $a = 10$, in Equation (1), we shall have the equation,

$$10^x = n \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

When n is made equal to 1, in Equation (2), the corresponding value of x is 0. When n is made equal to 10, the corresponding value of x is 1. When n is made equal to 100, the corresponding value of x is 2; and so on. Hence, we have, from what precedes,

$$\begin{aligned} \log \quad 1 &= 0, \\ \log \quad 10 &= 1, \\ \log \quad 100 &= 2, \\ \log \; 1000 &= 3, \text{ \&c.} \end{aligned}$$

For all values of n between 1 and 10, the corresponding logarithms lie between 0 and 1; that is, they are fractions less than 1, and are generally expressed decimally. For all values of n between 10 and 100, the corresponding logarithms lie between 1 and 2; that is, they are equal to 1 *plus* a decimal. The logarithms of all numbers between 100 and 1000, lie between 2 and 3; that is, they are equal to 2 *plus* a decimal. In general, a logarithm is composed of two parts: *an entire part*, called the *characteristic;* and a *decimal part*, sometimes called the *mantissa.*

Logarithms are often used to facilitate numerical computations, where they serve to convert operations of mul-

tiplication and division into the simpler ones of addition and subtraction. The following principles indicate the methods of applying logarithms to arithmetical computations.

GENERAL PRINCIPLES.

185. Let a denote the base of any system of logarithms, m and n any two numbers, and x and y their logarithms. We shall have, from Equation (1),

$$a^x = m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$
$$a^y = n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Multiplying (3) and (4), member by member, we have,

$$a^{x+y} = mn.$$

Whence, from the definition,

$$x + y = \text{Log } mn \quad . \quad . \quad . \quad . \quad (5.)$$

That is, *the logarithm of the product of two numbers is equal to the sum of the logarithms of the two numbers.*

If we divide (3) by (4), member by member, we shall have,

$$a^{x-y} = \frac{m}{n}.$$

Whence, from the definition,

$$x - y = \text{Log } \frac{m}{n} \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

That is, *the logarithm of the quotient is equal to the logarithm of the dividend diminished by that of the divisor.*

If we raise both members of (3) to any power denoted by p, we have,

$$a^{px} = m^p.$$

Whence, by definition,

$$px = \text{Log } m^p \quad . \quad . \quad . \quad (7.)$$

That is, *the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

If we extract any root of both members of (3), denoted by r, we have,

$$a^{\frac{x}{r}} = \sqrt[r]{m}.$$

Whence, by definition,

$$\frac{x}{r} = \text{Log } \sqrt[r]{m} \quad . \quad . \quad . \quad . \quad (8.)$$

That is, *the logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

The applications of the above principles require a table of logarithms. A TABLE OF LOGARITHMS, is a table by means of which the logarithm corresponding to any number, or the number corresponding to any logarithm, may be found.

The principles above demonstrated, give rise to four practical

RULES.

1. *To find the product of two or more numbers.*

Find the logarithms of the factors from a table, and take their sum; then find the number corresponding to the resulting logarithm, and it will be the product required.

2. *To find the quotient of one number by another.*

Find the logarithms of the dividend and divisor from a table, and subtract the latter from the former; then

find the number corresponding to the resulting logarithm, and it will be the quotient required.

3. *To raise a number to any power.*

Find the logarithm of the number from a table, and multiply it by the exponent; then find the number corresponding to the resulting logarithm, and it will be the power required.

4. *To extract any root of a number.*

Find the logarithm of the number from a table, and divide it by the index; then find the number corresponding to the resulting logarithm, and it will be the root required.

No practical examples can be given to illustrate the preceding rules, without a table of logarithms. A few examples of transformation are, however, annexed, which serve to show the methods of proceeding in the employment of logarithms.

EXAMPLES.

1. Transform the equation, $x = \frac{abc}{df}$.

From Equations (5) and (6), using common logarithms, we have,

$$\log x = \log a + \log b + \log c - \log d - \log f.$$

2. Transform the equation, $x = \frac{a^7 b^8}{\sqrt[3]{a^2}}$.

From Equations (5), (6), (7), and (8), we have,

$$\log x = 7 \log a + 8 \log b - \tfrac{2}{3} \log a = 6\tfrac{1}{3} \log a + 8 \log b.$$

3. Transform the equation, $x = \dfrac{a^{\frac{2}{3}}b^4c^{\frac{1}{3}}}{\sqrt[3]{fg}}$.

Ans. $\text{Log } x = \frac{2}{3}\log a + 4\log b + \frac{1}{3}\log c - \frac{1}{3}\log f - \frac{1}{3}\log g.$

4. Transform the equation, $x = \dfrac{\sqrt{5}\times\sqrt[3]{6}}{\sqrt[7]{3}\times\sqrt[9]{2}}$.

Ans. $\log x = \frac{1}{2}\log 5 + \frac{1}{3}\log 6 - \frac{1}{7}\log 3 - \frac{1}{9}\log 2.$

5 Transform the equation, $x = \dfrac{\sqrt[3]{(a^3-b^2)\times 3a}}{\sqrt{(a+b)c^3}}$.

Ans. $\log x = \frac{1}{3}\log(a-b) + \frac{1}{3}\log a + \frac{1}{3}\log 3 - \frac{1}{6}\log(a+b) - \frac{3}{2}\log c.$

6. Transform the equation, $x = \dfrac{a^3 - 2ba^2 + bc^2a}{a^2 - ba + 4cd}$.

Dividing both terms of the fraction by a, and factoring,

$$x = \frac{a\left(a - 2b + \dfrac{bc^2}{a}\right)}{a - b + \dfrac{4cd}{a}}.$$

Making $\dfrac{bc^2}{a} = h,$

whence, $\log h = \log b + 2\log c - \log a;$ . . . $(a.)$

and, $\dfrac{4cd}{a} = h',$

whence, $\log h' = \log 4 + \log c + \log d - \log a;$. $(b.)$

we find,

$$\log x = \log a + \log(a - 2b + h) - \log(a - b + h').$$

h and h', may be found from Equations (a) and (b)

and then the value of x may be found from the final equation.

7. Transform the equation,

$$x = \frac{2a^4 - 3ab^3 + b^4}{a^3 - 3a^2b + 4b^2c} = \frac{a\left(2a - \dfrac{3b^3}{a^2} + \dfrac{b^4}{a^3}\right)}{a - 3b + \dfrac{4b^2c}{a^2}}.$$

Making

$$\frac{3b^3}{a} = m, \text{ whence, } \log m = \log 3 + 3\log b - \log a; \quad (a.)$$

$$\frac{b^4}{a^3} = n, \text{ whence, } \log n = 4\log b - 3\log a; \quad . \quad . \quad (b.)$$

$$\frac{4b^2c}{a^2} = r, \log r = \log 4 + 2\log b + \log c - 2\log a; \quad (c.)$$

we find,

$$\log x = \log a + \log(2a - m + n) - \log(a - 3b + r).$$

m, n, and r, may be found from Equations (a), (b), and (c), and then x may be computed from the final equation.

8. Transform the equation, $x = \sqrt{a \times \sqrt{b \times \sqrt{c}}}$.

Ans. $\log x = \frac{1}{2}\log a + \frac{1}{4}\log b + \frac{1}{8}\log c$.

The General Principles serve also to solve exponential equations, that is, equations in which the unknown quantities enter an exponent.

Solve the following equations:

9. $7^x = 13$.

Taking the logarithms of both members, we have,

$$x \log 7 = \log 13. \qquad \therefore \; x = \frac{\log 13}{\log 7}. \quad \textit{Ans.}$$

10. $\left(\frac{5}{7}\right)^x = \frac{2}{3}$. *Ans.* $x = \dfrac{\log 2 - \log 3}{\log 5 - \log 7}$.

11. $ab^x = c$. *Ans.* $x = \dfrac{\log c - \log a}{\log b}$.

12. $3^{\sqrt{x}} = 5$. *Ans.* $\left(\dfrac{\log 5}{\log 3}\right)^2$.

13. $a^{b^x} = c$.

Making

$b^x = d$; whence, $a^d = c$, or $d = \dfrac{\log c}{\log a}$, and $x = \dfrac{\log d}{\log b}$.

First find the value of d, then that of x.

14. $\left.\begin{aligned} a^x b^y &= c \\ my &= nx \end{aligned}\right\}$

Taking the logarithms of both members of the first equation, we have,

$$x \log a + y \log b = \log c.$$

Combining this with the second of the given equations, we find,

$$x = \frac{m \log c}{m \log a + n \log b}, \quad \text{and} \quad y = \frac{n \log c}{m \log a + n \log b}.$$

15. $(a^2 - b^2)^{2(x-1)} = (a - b)^{2x}$.

Ans. $x = 1 + \dfrac{\log (a - b)}{\log (a + b)}$.

16. $\left.\begin{aligned} 2^x 3^y &= 2000 \\ 5x - 3y &= 0 \end{aligned}\right\}$

Taking the logarithms of both members of the first equation, remembering that the logarithm of 2000 is equal to the logarithm of 1000×2, equal to $3 + \log 2$, we have,

$$x \log 2 + y \log 3 = 3 + \log 2;$$

which, combined with the second of the given equations, gives,

$$x = \frac{3(3 + \log 2)}{3 \log 2 + 5 \log 3}, \quad \text{and} \quad y = \frac{5(3 + \log 2)}{3 \log 2 + 5 \log 3}.$$

17. $2^x = \sqrt{2 \times \sqrt{3 \times \sqrt{5}}}$.

Ans. $x = \dfrac{\frac{1}{2} \log 2 + \frac{1}{4} \log 3 + \frac{1}{8} \log 5}{\log 2}$

18. $5^{x^3} = 7$. *Ans.* $x = \sqrt[3]{\dfrac{\log 7}{\log 5}}$.

GENERAL PROPERTIES OF LOGARITHMS.

186. There are certain general properties of logarithms that may be discovered by a discussion of the exponential equation,

$$a^x = n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

In this equation, the arbitrary quantities are a and n.

First. If we make $n = 1$, the corresponding value of x will be 0, whatever may be the value of a, since $a^0 = 1$, (Art. 31). Hence,

The logarithm of 1, *in any system whatever, is equal to* 0.

Second. If we make $n = a$, the corresponding value of x will be 1, whatever may be the value of a. Hence,

The logarithm of the base of any system, taken in that system, is 1.

Third. If we suppose $a > 1$, say 10, for example, we shall have,

$$10^x = n.$$

When $n = 1$, the value of x, or the logarithm of 1, is 0; when $n = \infty$, the value of x, or the logarithm of ∞, is ∞. The logarithms of all numbers between

1 and ∞, lie between 0 and ∞, that is, they are positive.

When n is less than 1, x must be negative, giving $\frac{1}{10^x} = n$; when $n = 0$, x will be infinite, in the last equation, because $\frac{1}{10^\infty} = 0$, therefore, $x = -\infty$, in the given equation, that is, the logarithm of 0 is equal to $-\infty$. Hence,

In any system whose base is greater than 1, *the logarithms of all numbers greater than* 1, *are positive; the logarithms of all numbers less than* 1, *are negative; the logarithm of* ∞, *is* $+\infty$, *and the logarithm of* 0, *is* $-\infty$.

Fourth. If we suppose $a < 1$, say $\frac{1}{10}$, for example, we shall have,

$$\frac{1}{10^x} = n, \text{ or } 10^{-x} = n.$$

In this case, the positive values of x correspond to the negative values of x in the preceding case; and the negative values of x, to the positive values, in the preceding case. Hence,

In any system whose base is less than 1, *the logarithms of all numbers greater than* 1, *are negative; the logarithms of all numbers less than* 1, *are positive; the logarithm of* ∞, *is* $-\infty$, *and the logarithm of* 0, *is* $+\infty$.

Fifth. Since, for every value of x between $-\infty$ and $+\infty$, that is, for every real value of x, the values of n lie between 0 and $+\infty$, whether a is greater or less than 1, it follows that there are no real values of x, which, substituted in the equation, $a^x = n$, will make n negative. Hence,

There are no real logarithms corresponding to negative numbers.

Although there are no logarithms of negative numbers, we may multiply negative numbers by means of their logarithms. We must regard the numbers as positive; and, having applied the rules, we must give the proper sign to the result, according to the rule of signs. Thus, to multiply 27 by -435, we find the product of 27 and 435, and give to it the minus sign, $-$.

CHAPTER XII.

GENERAL THEORY OF EQUATIONS.

187. EVERY equation containing but a single unknown quantity, whose exponents are whole numbers, may be reduced to the form,

$$x^n + px^{n-1} + qx^{n-2} + \&c. + sx^2 + tx + u = 0 \quad (1.)$$

In this equation, n is a positive whole number, but the coefficients p, q, &c., may be either positive or negative, entire or fractional, real or imaginary. The method of reducing equations to the form (1), is entirely analogous to that given for reducing equations of the second degree to the form,

$$x^2 + 2px = q;$$

and since the reduction can always be made, we shall hereafter, in speaking of equations, suppose them reduced to the form of Equation (1), unless the contrary is expressly mentioned.

188. Any value of x, either real or imaginary, which when substituted for x, in Equation (1), will satisfy it, that is, make the two members equal, is a *root* of the equation.

It is evident, that Equation (1) has at least one root, for if the first member is equal to 0, it must be so for some value of x, either real or imaginary, and this value of x is, by definition, a root.

PROPERTIES AND TRANSFORMATIONS OF EQUATIONS.

I. *In every equation of the form* (1), *if* a *is a root, the first member is divisible by* $x - a$.

189. For, by applying the rule for division, and continuing the operation till a remainder is found which does not contain x, denoting the quotient by m and the remainder by n, we shall have,

$$x^n + px^{n-1} + \&c. + tx + u = (x-a)m + n \quad (2.)$$

Now, if a is a root of the proposed equation, it will reduce the first member of (2) to 0, when substituted for x; it will also reduce the first term of the second member to 0; hence, n is also equal to 0, that is, the remainder is 0, and consequently the first member is exactly divisible by $x - a$.

II. Conversely, *if the first member of an equation of the form* (1) *is exactly divisible by* $x - a$, *then is* a *a root of the equation.*

190. For, in this case, the remainder n is equal to 0, which reduces Equation (2) to the form,

$$x^n + px^{n-1} + \&c. + tx + u = (x - a)m \quad (3.)$$

If, in (3), we make $x = a$, the second member reduces to 0; consequently, the first member also reduces to 0, which satisfies Equation (1). Hence, a is a root of (1), as was enunciated.

It follows, from the preceding propositions, that we can ascertain whether a polynomial containing x is exactly divisible by a binomial of the form, $x - a$, by simply

substituting a for x in the polynomial. If the result is 0, the polynomial is divisible by $x - a$; if it is not 0, the polynomial is not exactly divisible by $x - a$.

III. *Every equation of the form* (1), *has as many roots as there are units in* n, *and no more.*

191. Since the equation has at least one root, let that root be denoted by a, then will $x - a$ be a factor of the first member; the first term of the other factor is x^{n-1}, and the exponent of x, in each of the other terms, is less than $n - 1$; hence, the given equation may be written under the form,

$$(x - a)(x^{n-1} + p'x^{n-2} + \&c. + t'x + u') = 0 \quad . \quad (4.)$$

Now, Equation (4) may be satisfied in two ways, viz.: by placing the first factor equal to 0, or by placing the second factor equal to 0. In the latter case, we shall have the equation,

$$x^{n-1} + p'x^{n-2} + \&c. + t'x + u' = 0 \quad . \quad . \quad (5.)$$

Now, Equation (5) has at least one root; let that root be designated by b; then it may be shown as before, that Equation (5) can be written under the form,

$$(x - b)(x^{n-2} + p''x^{n-3} + \&c. + t''x + u'') = 0; \quad (6.)$$

which reduces Equation (4) to the form of,

$$(x-a)(x-b)(x^{n-2}+p''x^{n-3}+ \&c. + t''x+u'') = 0 \quad (7.)$$

In the same manner as before, it may be shown that the second factor of the first member of (6) can be placed equal to 0, and factored, giving,

$$(x \quad c)(x^{n-3} + p'''x^{n-4} + \&c. + t'''x + u''') = 0;$$

which reduces Equation (4) to the form,

$$(x-a)(x-b)(x-c)(x^{n-3}+p'''x^{n-4}+\&c.+t'''x+u''') = 0.$$

By continuing the process, it may be shown that the first member will ultimately be resolved into just as many binomial factors, of the form, $(x-a)$, $(x-b)$, &c., as there are units in n. Hence, Equation (1) may be written under the form,

$$(x-a)(x-b)(x-c)\ldots(x-k)(x-l) = 0. \quad . \quad . \quad (8.)$$

Equation (8) may be satisfied, by placing either of the factors, $x-a$, $x-b$, &c., equal to 0, and either factor being placed equal to 0, gives a root. Now, since there are n factors, the equation has n roots, and since the equation cannot be satisfied except by making one of the factors equal to 0, there are only n roots: which was to be shown.

If both members of Equation (8), be divided by either of the factors, $x-a$, $x-b$, &c., it will be reduced in degree by 1. It is to be observed, that some of the roots, and consequently, some of the binomial factors, may be equal to each other.

EXAMPLES.

1. One root of the equation, $x^3-9x^2+26x-24=0$, is 4; what does the equation become when freed of this root?

OPERATION.

$$\begin{array}{r|l} x^3-9x^2+26x-24 & x-4 \\ \underline{x^3-4x^2} & \overline{x^2-5x+6} \\ -5x^2+26x & \\ \underline{-5x^2+20x} & \\ 6x-24 & \\ \underline{6x-24} & \\ 0 & \end{array}$$

Hence, the required equation is,

$$x^2 - 5x + 6 = 0,$$

an equation that may be solved by known rules.

2. One root of the equation, $x^3 - 37x - 84 = 0$, is $+7$; what does it become when freed of this root?

Ans. $x^2 + 7x + 12 = 0$.

3. One root of the equation, $x^3 - 11x^2 + 16x + 84 = 0$, is -2; what does the equation become when freed of this root? *Ans.* $x^2 - 13x + 42 = 0$.

4. One root of the equation, $x^3 + 7x^2 - 4x - 28 = 0$, is -7; what does the equation become when freed of this root? *Ans.* $x^2 - 4 = 0$.

5. One root of the equation, $x^3 - 12x^2 + 47x - 60 = 0$, is 3; what are the other roots? *Ans.* 4 and 5.

6. One of the roots of $x^3 + 9x^2 + 26x + 24 = 0$, is -4; what are the other two? *Ans.* -2 and -3.

7. Two roots of the equation, $x^4 - 12x^3 + 48x^2 - 68x + 15 = 0$, are 3 and 5; what are the other two?

Ans. $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

8. One root ot the equation, $x^3 - 6x^2 + 11x - 6 = 0$, is 1; what are the other two. *Ans.* 3 and 2.

192. Equation (8) indicates the method of forming an equation whose roots are given.

RULE.

Subtract each root from the unknown quantities; multiply the resulting binomials together, and place the product equal to 0.

EXAMPLES.

1. Find the equation whose roots are 1, 2, and 3.

Subtracting each root from x, we have the binomial factors,

$$x - 1, \quad x - 2, \quad \text{and} \quad x - 3.$$

Multiplying them together, and placing the product equal to 0, we find,

$$x^3 - 6x^2 + 11x - 6 = 0,$$

which is the required equation.

2. Find the equation whose roots are -7 and -4.

Ans. $x^2 + 11x + 28 = 0.$

3. Find the equation whose roots are 3, 4, and -7.

Ans. $x^3 - 37x + 84 = 0.$

4. Find the equation whose roots are -1, -4, and -8. *Ans.* $x^3 + 13x^2 + 44x + 32 = 0.$

5. Find the equation whose roots are -2, -2, $+4$, and -4. *Ans.* $x^4 + 4x^3 - 12x^2 - 64x - 64 = 0.$

6. Find the equation whose roots are equal to -3, -3, and -3. *Ans.* $x^3 + 9x^2 + 27x + 27 = 0.$

7. Find the equation whose roots are 2, 3, 5, and -6. *Ans.* $x^4 - 4x^3 - 29x^2 + 156x - 180 = 0.$

8. Find the equation whose roots are 1, 2, and -5.

Ans. $x^3 + 2x^2 - 13x + 10 = 0.$

9. Find the equation whose roots are 3, 4, -1, and -6. *Ans.* $x^4 - 31x^2 + 42x + 72 = 0.$

10. Find the equation whose roots are -3, -4, $2 + 3\sqrt{-1}$, and $2 - 3\sqrt{-1}$.

Ans. $x^4 + 3x^3 - 3x^2 + 43x + 156 = 0.$

193. The above results have been obtained by the process of actual multiplication of the binomial factors. There is, however, a simple law, by means of which the coefficients of the different powers of the unknown quantity may be found. To demonstrate the law, let a, b, $c \ldots k$, l, represent the roots of an equation of the n^{th} degree. Forming the factors, according to the rule, we have,

$$(x - a)\ (x - b)\ (x - c) \ldots (x - k)\ (x - l);$$

If we perform the multiplication as far as three factors, we shall have the result shown below:

$$\begin{array}{l} x - a \\ x - b \\ \hline x^2 - a \mid x + ab \\ \quad - b \mid \end{array}$$

$$\begin{array}{l} x - c \\ \hline x^3 - a \mid x^2 + ac \mid x - abc \\ \quad - b \mid \quad + bc \mid \\ \quad - c \mid \quad + ab \mid \end{array}$$

Examining this product of three factors, we see that it is subject to two simple laws of formation.

1. *The exponent of x, in the first term, is equal to the number of factors employed, and it goes on diminishing by* 1, *in each term, to the right, to the last term, where it is* 0.

2. *The coefficient of the first term is* 1; *that of the second term is equal to the algebraic sum of the roots with their signs changed; that of the third term is equal to the algebraic sum of the different products of the roots*

with their signs changed, taken in sets of 2; *and so to the coefficient of the last term, which is equal to the continued product of the roots with their signs changed.*

By a process entirely analogous to that employed in demonstrating the Binomial Formula, which is unnecessary to repeat in this place, it may be shown that, *if the laws above enunciated hold true for the formation of the product of m binomial factors, they will also hold true for the formation of the product of* $m + 1$ *such factors.*

But the laws for the formation of the product of three factors have been shown true by actual multiplication; hence, they are true for the formation of the product of four factors. Being true for four factors, they must be true for five. Being true for five, they must be so for six; and so on, by successive deduction, to any desired extent: hence, they are true for any number of factors. Hence, we have the following rule for forming an equation when its roots are given.

RULE.

I. *The exponent of the unknown quantity in the first term is equal to the number of roots, and goes on diminishing by* 1, *in each term, to the right, to the last term, where it is* 0.

II. *The coefficient of the first term is* 1. *To find the remaining coefficients, change the signs of all the roots; find the algebraic sum of the results, and it will be the coefficient of the second term; find the algebraic sum of the different products of the results taken two in a set, and it will be the coefficient of the third term; find the algebraic sum of the different products of the results taken three in a set, and it*

will be the coefficient of the fourth term; proceed in this way, to the last term, which will be equal to the continued product of all the results.

The last term, containing the 0 power of the unknown quantity, is called the *absolute term.*

194. If the sum of the positive roots is equal to the sum of the negative roots, numerically considered, their algebraic sum will be 0, and consequently the coefficient of the second term of the equation will be 0, and that term will disappear from the equation; conversely, if the second term is wanting, the sum of the positive roots is equal to the sum of the negative roots.

EXAMPLES.

1. Find the equation whose roots are 1, 5, and 9.

The coefficient of x^3, is 1. The coefficient of x^2, is $-1-5-9 = -14$. The coefficient of x, is $(-1 \times -5) + (-1 \times -9) + (-5 \times -9) = +59$, and the absolute term is, $-1 \times -5 \times -9 = -45$; hence, the required equation is,

$$x^3 - 14x^2 + 59x - 45 = 0.$$

2. Find the equation whose roots are -3, -4, -5, and -6.

The coefficient of x^4, is 1. The coefficient of x^3, is $3 + 4 + 5 + 6 = 18$. The coefficient of x^2, is $3 \times 4 + 3 \times 5 + 3 \times 6 + 4 \times 5 + 4 \times 6 + 5 \times 6 = 119$. The coefficient of x, is $3 \times 4 \times 5 + 3 \times 4 \times 6 + 3 \times 5 \times 6 + 4 \times 5 \times 6 = 342$. The absolute term is, $3 \times 4 \times 5 \times 6 = 360$; hence, the required equation is,

$$x^4 + 18x^3 + 342x + 360 = 0.$$

3. Find the equation whose roots are -1, -2, and 3.

Ans. $x^3 - 7x - 6 = 0.$

4. Find the equation whose roots are $-\frac{1}{2}$, $-\frac{1}{4}$, and $-\frac{1}{8}$.

Ans. $x^3 + \frac{7}{8}x^2 + \frac{7}{32}x + \frac{1}{64} = 0.$

5. Find the equation whose roots are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$.

Ans. $x^3 - \frac{31}{30}x^2 + \frac{1}{3}x - \frac{1}{30} = 0.$

The law of formation above illustrated, is of much use in investigating the properties of equations and their roots, as will be seen hereafter.

IV. *Every equation of the form* (1), *whose coefficients are fractional, can be reduced to another equation of the same form, whose coefficients are entire.*

195. Take the equation,

$$x^3 + \frac{7}{8}x^2 + \frac{7}{32}x + \frac{1}{64} = 0.$$

Substitute for x, the fraction $\frac{y}{k}$, in which y is a new unknown quantity, and k an arbitrary quantity. We shall have,

$$\frac{y^3}{k^3} + \frac{7}{8}\cdot\frac{y^2}{k^2} + \frac{7}{32}\cdot\frac{y}{k} + \frac{1}{64} = 0.$$

Multiplying both members by k^3, we have,

$$y^3 + \frac{7k}{8}y^2 + \frac{7k^2}{32}y + \frac{k^3}{64} = 0.$$

Since k is arbitrary, we may give to it such a value as will make the coefficients, $\frac{7k}{8}$, $\frac{7k^2}{32}$, and $\frac{k^3}{64}$, whole numbers. This can always be done; because, if we

make k equal to the least common multiple of the denominators, the different powers of k will be divisible by the denominators separately, and consequently, the coefficients will be whole numbers. Making $k = 64$, the last equation becomes,

$$y^3 + 56y^2 + 896y + 3096 = 0.$$

It often happens, that a less value of k will make the coefficients entire, as in the above example. Resolving the denominators into factors, the coefficients become,

$$\frac{7k}{2.2.2}, \quad \frac{7k^2}{2.2.2.2.2}, \quad \frac{k^3}{2.2.2.2.2.2}.$$

We see that $k = 2.2.2$, will render all of these entire, giving,

7, 14, and 16.

Hence, the transformed equation is,

$$y^3 + 7y^2 + 14y + 16 = 0.$$

If the roots of the last equation were known, those of the given equation might be found, by dividing each by 8, since $x = \frac{y}{8}$.

EXAMPLES.

1. Transform the equation, $x^3 - \frac{7}{3}x^2 + \frac{11}{36}x - \frac{25}{72} = 0$, into one whose coefficients are entire, that of the first term being 1.

Making $x = \frac{y}{k}$, and multiplying both members by k^3,

$$y^3 - \frac{7k}{3} + \frac{11k^2}{36} - \frac{25k^3}{72} = 0.$$

The fractional coefficients, when their denominators are factored, are,

$$-\frac{7k}{3}, \quad \frac{11k^2}{2.2.3.3}, \quad \text{and} \quad -\frac{25k^3}{2.2.2.3.3}.$$

Making $k = 2 \times 3 = 6$, these coefficients become,

$$-14, \quad 11, \quad \text{and} \quad -75.$$

Hence, the transformed equation required is,

$$y^3 - 14y + 11y - 75 = 0.$$

The roots of this equation, divided by 6, give the roots of the assumed equation.

2. Transform $x^3 - \frac{41}{30}x^2 + \frac{1}{3}x - \frac{1}{30} = 0$, into one whose coefficients are entire.

Ans. $y^3 - 41y^2 + 300y - 900 = 0.$

3. Transform the equation, $x^3 - \frac{2}{25}x^2 + \frac{1}{30}x - \frac{1}{40} = 0$, into one having entire coefficients.

Ans. $y^3 - 12y^2 + 750y - 84375 = 0.$

4. Transform the equation, $x^3 + \frac{1}{4}x^2 + \frac{1}{16}x + \frac{1}{64} = 0$, into one whose coefficients are whole numbers.

Ans. $y^3 + y^2 + y + 1 = 0.$

V. *Any equation of the form* (1), *may be transformed into another of the same form, in which the roots are any given multiple of those of Equation* (1).

196. For, substituting in (1), $\frac{y}{k}$ for x, and multiplying through by k^n, we have,

$$y^n + kpy^{n-1} + k^2qy^{n-2} + \ldots + k^{n-1}ty + k^n u = 0; \quad (9.)$$

an equation of the proposed form, having its roots k times as great as those of (1), since $y = kx$ by hypothesis; k may be entire or fractional.

EXAMPLES.

1. Transform the equation, $x^2 - 7x + 12 = 0$, into another in which the roots are twice as great.

Then, $x = \frac{y}{2}$, and the resulting equation is,

$$\frac{y^2}{4} - \frac{7y}{2} + 12 = 0; \text{ or, } y^2 - 14y + 48 = 0.$$

2. Transform the equation, $x^2 - 4x - 21 = 0$, into another whose roots are three times as great.

Ans. $y^2 - 12y - 189 = 0.$

3. Transform the equation, $x^2 - 4x - 32 = 0$, into one whose roots are half as great.

Ans. $y^2 - 2y - 8 = 0.$

4. Transform the equation, $x^2 + 11x + 28 = 0$, into one whose roots are twice as great.

Ans. $y^3 + 22y + 112 = 0.$

5. Transform the equation, $x^4 + 4x^3 - 12x^2 - 64x - 64 = 0$, into one whose roots are half as great.

Ans. $y^4 + 2y^3 - 3y^2 - 8y - 4 = 0.$

VI. *Any equation of the form* (1), *may be reduced to another equation of the same form, whose roots differ from those of the given equation by any given quantity.*

197. For, making in Equation (1), $x = y + r$, in which y is a new unknown quantity, and r an arbitrary quantity, we have,

$$(y + r)^n + p(y + r)^{n-1} + \ldots t(y + r) + u = 0.$$

Developing the different powers of the binomial, $y + r$, by the Binomial Formula, arranging according to the powers of y, and denoting the coefficients of the $n - 1$, $n - 2$, &c., powers of y, by p', q', &c., we have,

$$y^n + p'y^{n-1} + q'y^{n-2} + \ldots + s'y^2 + t'y + u' = 0, \quad (a).$$

an equation of the proposed form, whose roots differ from those of (1) by r, since $y = x - r$, by hypothesis.

The operation of making this transformation is somewhat tedious, but a simple rule may be deduced for finding the coefficients of the transformed equation, that will render the transformation sufficiently expeditious.

Equation (a) was derived from (1), by making $x = y + r$; hence, if we make $y = x - r$, in Equation (a), the resulting equation will be identical with Equation (1). Making this substitution, we have,

$$(x - r)^n + p'(x - r)^{n-1} + \ldots + s'(x - r)^2 + t'(x - r) + u' = 0. \quad \ldots \quad (b.)$$

Now, if the first member of (b), or what is the same thing, the first member of Equation (1), be divided by $x - r$, the remainder will be equal to u', the coefficient of y^0, in Equation (a). The quotient obtained is,

$$(x - r)^{n-1} + p'(x - r)^{n-2} + s'(x - r) + t'.$$

If this quotient be again divided by $x - r$, the remainder will be t', the coefficient of y, in Equation (a). If the second quotient be divided by $x - r$, the remainder will be the coefficient of y^2 in Equation (a); and

so on. By continuing the process of successive division, the successive remainders will be the coefficients of the transformed equation, taken in a reverse order.

EXAMPLES.

1. Find an equation whose roots shall be less by 2, than those of the equation, $x^3 - 6x^2 - 7x + 60 = 0$.

OPERATION.

$$
\begin{array}{l|l|l|l}
x^3 - 6x^2 - 7x + 60 & x - 2 & & \\
\underline{x^3 - 2x^2} & \underline{x^2 - 4x - 15} & x - 2 & \\
\quad - 4x^2 - 7x & x^2 - 2x & \underline{x - 2} & x - 2 \\
\quad \underline{4x^2 + 8x} & \quad - 2x - 15 & x - 2 & 1 \\
\quad\quad -15x + 60 & \quad - 2x + 4 & 0 & \\
\quad\quad \underline{-15x + 30} & \quad\quad - 19 & & \\
\quad\quad\quad + 30 & & &
\end{array}
$$

Here, we divide $x^3 - 64^2 - 7x + 60$, by $x - 2$, and find a quotient, $x^2 - 4x - 15$, with a remainder, $+ 30$. We next divide $x^2 - 4x - 15$, by $x - 2$, and find a quotient, $x - 2$, with a remainder, $- 19$. We next divide $x - 2$, by $x - 2$, and find a quotient, 1, with a remainder, 0. Hence, the transformed equation is,

$$y^3 - 19y + 30 = 0.$$

The roots of the given equation are, 4, 5, and $- 3$, and those of the transformed equation are, 2, 3, and $- 5$, as may be shown by forming the equations in the two cases.

2. Find the equation whose roots are greater by 1, than those of the equation, $x^3 + 4x^2 + 5x + 7 = 0$.

OPERATION.

$$
\begin{array}{l|l}
x^3 + 4x^2 + 5x + 7 & x + 1 \\
\underline{x^3 + x^2} & \overline{x^2 + 3x + 2} \\
3x^2 + 5x & x^2 + x \\
\underline{3x^2 + 3x} & \overline{2x + 2} \\
2x + 7 & 2x + 2 \\
\underline{2x + 2} & \overline{0} \\
5 &
\end{array}
\qquad
\begin{array}{l|l}
x + 1 & \\
\overline{x + 2} & x + 1 \\
x + 1 & 1 \\
\overline{1} &
\end{array}
$$

Hence, the equation required is,

$$y^3 + y^2 + 5 = 0.$$

3. Find the equation whose roots are less by 1, than those of the equation, $x^3 - 7x + 7 = 0$.

Ans. $y^3 + 3y^2 - 4y + 1 = 0.$

4. Find the equation whose roots are greater by 10, than those of the equation, $x^4 + 42x^3 + 663x^2 + 4664x = 0$.

Ans. $y^4 + 2y^3 + 3y^2 + 4y - 12340 = 0.$

198. The operation of successive division may be greatly abridged by the method of synthetic division, which will now be explained, as far as necessary for our purpose.

If a polynomial of the form,

$$x^n + px^{n-1} + qx^{n-2} + \ldots sx^2 + tx + u,$$

be divided by a binomial of the form, $x - r$, or $x + r$, the quotient will be of the form,

$$x^{n-1} + p'x^{n-2} + \&c. + s'x + t,$$

and the remainder will be independent of x.

Now, as our object is to find the remainder, we need

only consider the coefficients in the dividend, divisor, and quotient. To illustrate, let us take Example 1. Writing down the coefficients only, we have,

OPERATION.

$$\begin{array}{r|l} 1-6-7+60 & 1-2 \\ 1-2 & \hline 1-4-15 \\ \hline -4-7 & 1-2 \\ -4+8 & \hline -2-15 \\ \hline -15+60 & -2+4 \\ -15+30 & \hline -19 \\ \hline +30 & \end{array} \qquad \begin{array}{r|l} 1-2 & \\ \hline 1-2 & 1-2 \\ \hline 0 & \end{array}$$

Here, we have found the first remainder, 30; and the quotient might be obtained by annexing the successive powers of x, from x^2 down to x^0, to the several coefficients of the quotient; but, for our purpose, this quotient is not needed; for the process may be continued, as indicated above, till all of the remainders are found.

This operation may be abbreviated; for, on examining the process, we see that the first term of the quotient in each division is 1; the second term of the quotient is obtained by multiplying the first term by the second term of the divisor, and subtracting the result from the second term of the dividend; or, what is the same thing, by multiplying the first term of the quotient by the second term of the divisor with its sign changed, and adding the result to the second term of the dividend; the third term of the quotient is found by multiplying the second term by the second term of the divisor with its sign changed, and adding the result to the third term of the dividend; and so on, till the last term of the dividend has been used: the last result is the remainder required.

The above process, when written out, takes the form,

$$\begin{array}{lll} & 1-6-7+60 \mid +2 & \\ & \quad +2-8-30 & \\ \hline 1st\ quotient, & 1-4-15,+30 & 1st\ remainder. \\ & \quad +2-4 & \\ \hline 2d\ quotient, & 1-2,-19 & 2d\ remainder. \\ & \quad +2 & \\ \hline 3d\ quotient, & 1,+0 & 3d\ remainder. \end{array}$$

Here, we have changed the sign of the second term of the divisor, and dropped the first term entirely. The first term of the first quotient is 1; multiplying 1 by 2, we find 2, which we write under -6, and adding -6 and $+2$, we find -4, which we write for the second term of the quotient; multiplying -4 by 2, we find -8, which we write under -7, and adding -7 and -8, we get -15, which we write for the third term of the quotient; multiplying -15 by 2, we get -30, which we place under 60, and adding, we get 30 for the first remainder, which we point off by a comma.

Commencing the second division, we write 1 for the first term of the quotient; multiplying by 2, and adding to -4, we get -2, for the second term; multiplying this by 2, and adding to -15, we get -19 for the second remainder.

Commencing the third division, we write 1 for the first term of the quotient; multiplying by 2, and adding to -2, we get 0 for the third remainder.

In the same way, all similar divisions may be performed, as shown in the following examples:

5. Find an equation whose roots are greater by 3 than those of the equation,

$$x^4 - 17x^2 + 12x^2 - 33x + 67 = 0.$$

Here, the divisor is $x + 3$, and the second term, with its sign changed, is -3.

OPERATION.

1	-17	$+12$	-33	$+67$	$\vert -3.$												
	-3	$+60$	-216	$+747$													
1	-20	$+72$	$-249,$	$+814$	. . .	1*st rem.*											
	-3	$+69$	-423														
1	-23	$+141,$	-672	. . . . .	2*d rem.*												
	-3	$+78$															
1	$-26,$	$+219$	. . . . .	3*d rem.*													
	-3																
1,	-29	. . . . .	4*th rem.*														

Hence, the equation required is,

$$y^4 - 29y^3 + 219y^2 - 672y + 814 = 0.$$

It is not necessary that the coefficient of the first term of the dividend should be 1. It may be any number, and in that case the first term of each quotient will be the same number.

6. Find the equation whose roots are less by 3, than those of the equation,

$$3x^4 - 4x^3 + 7x^2 + 8x - 12 = 0.$$

OPERATION.

3	-4	$+7$	$+8$	-12	$\vert +3$												
	$+9$	$+15$	$+66$	$+222$													
3	$+5$	$+22$	$+74,$	$+210$	. .	1*st rem.*											
	$+9$	$+42$	$+192$														
3	$+14$	$+64,$	$+266$	. . . . .	2*d rem.*												
	$+9$	$+69$															
3	$+23,$	$+133$	. . . . .	3*d rem.*													
	$+9$																
3,	$+32$	. . . . .	4*th rem.*														

Hence, the required equation is,

$$3y^4 + 32y^3 + 133y^2 + 266y + 210 = 0.$$

When any terms are wanting in the dividend, their place must be supplied by 0's.

7. Find an equation whose roots are greater by 3, than those of the equation, $x^4 - 2 = 0$.

OPERATION.

$1 + 0 + 0 + 0 - 2 \mid -3$	
$-3 + 9 - 27 + 81$	
$1 - 3 + 9 - 27, + 79$ . . .	1*st rem.*
$-3 + 18 - 81$	
$1 - 6 + 27, - 108$	2*d rem.*
$-3 + 27$	
$1 - 9, + 54$	3*d rem.*
-3	
$1, - 12$	4*th rem.*

Hence, the required equation is,

$$y^4 - 12y^3 + 54y^2 - 108y + 79 = 0.$$

8. Find an equation whose roots are less by 0.1, than those of the equation, $x^3 - 1 = 0$.

OPERATION.

$1 + 0 + 0 - 1 \mid 0.1$	
$0.1 + 0.01 + 0.001$	
$1 + 0.1 + 0.01, - 0.999$	1*st rem.*
$+0.1 + 0.02$	
$1 + 0.2, + 0.03$	2*d rem.*
$+0.1$	
$1, + 0.3$	3*d rem.*

Hence, the required equation is,

$$y^3 + 0.3y^2 + 0.03y - 0.999 = 0.$$

9. Find an equation whose roots are greater by 3, than those of the equation, $x^4 + 13x^3 + x^2 - 11 = 0$.

Ans. $y^4 + y^3 - 62y^2 + 237y - 272 = 0.$

10. Find an equation whose roots are less by 2, than those of the equation, $x^4 - 9x^3 + 20x^2 - 10x - 1 = 0$.

Ans. $y^4 - y^3 - 10y^2 - 6y + 3 = 0.$

If the second term of the divisor used is equal to the quotient of the coefficient of the second term of the dividend with its sign changed, by the exponent denoting the degree of the dividend, the last remainder will, from the nature of the operations performed, be equal to 0, and the second term of the resulting equation will be 0, or *wanting.*

16. Transform the equation, $x^3 - 6x^2 + 7x - 10 = 0$, so that the resulting equation shall want the second term.

OPERATION.

1	-6	$+7$	-10	2	
	$+2$	-8	-2		
1	-4	$-1,$	-12	. . . . .	1*st rem.*
	$+2$	-4			
1	$-2,$	-5		. . . . .	2*d rem.*
	$+2$				
1,	$+0$			. . . . .	3*d rem.*

Hence, the required equation is,

$$y^3 - 5y - 12 = 0.$$

The roots of the resulting equation are less by 2, than those of the given equation.

Transform the following equations, so that the resulting equations shall want their second terms:

17. $x^3 + 9x^2 - x + 4 = 0$. *Ans.* $y^3 - 28y + 61 = 0$.

The roots of the resulting equation are less by 3, than those of the given equation.

18. $x^4 - 8x^3 + 7x^2 + 3x + 4 = 0$.

Ans. $y^4 - 17y^2 - 33y - 10 = 0$.

VII. *Any equation of the form* (1), *may be transformed into another equation of the same form whose roots are equal those of the given equation, with their signs changed.*

199. For, if in (1), we make $x = -y$, there will result the equation,

$$(-y)^n + p(-y)^{n-1} + \&c. + t(-y) + u = 0; \quad (10.)$$

which is of the proposed form. The following examples show what changes will result in making the proposed transformation.

First, when the degree of the equation is even, as,

$$x^4 + 2x^3 - 7x^2 + x - 8 = 0.$$

Making $x = -y$, we have,

$$y^4 - 2x^3 - 7y^2 - y - 8 = 0.$$

Here, the coefficients of the terms remain numerically the same as before, but the signs of the coefficients of the odd powers are changed. The same will hold true in all equations of an even degree, when thus transformed.

Second, when the degree of the equation is odd, as,

$$x^5 - 7x^4 + x^3 - 2x^2 - x - 1 = 0.$$

Making $x = -y$, and dividing both members by -1, we have,

$$y^5 + 7y^4 + y^3 + 2y - y + 1 = 0.$$

Here, the numerical values of the coefficients remain the same, but the signs of the coefficients of the even powers are changed. The same will hold true in all equations of an odd degree, when thus transformed.

In both cases, the transformation may be made by changing the signs of the second, fourth, sixth, &c., terms. We have supposed the equation complete; when it is incomplete, the wanting terms must be supplied by 0's, in making the transformation by the last rule.

VIII. *Any equation of the form* (1), *may be transformed into another equation of the same form, whose roots shall be equal to the reciprocals of those of the given equation.*

200. For, substituting $\frac{1}{y}$ for x, in Equation (1), we have,

$$\frac{1}{y^n} + \frac{p}{y^{n-1}} + \frac{q}{y^{n-2}} + \ldots + \frac{s}{y^2} + \frac{t}{y} + u = 0.$$

Multiplying both members of this equation, by $\frac{y^n}{u}$ and writing the terms in an inverse order, we have,

$$y^n + \frac{t}{u}y^{n-1} + \frac{s}{u}y^{n-2} + \ldots + \frac{q}{u}y^2 + \frac{p}{u}y + \frac{t}{u} = 0; \quad (11.)$$

which is of the required form. The coefficients are found by writing those of the given equation in an inverse order, and dividing each by the absolute term. When any terms are wanting in the given equation,

the one at the same distance from the other extreme will be wanting in the required equation.

EXAMPLES.

Transform the following equations into others whose roots shall be the reciprocals of those of the given equations.

1. $x^3 + 3x^2 + 9x + 3 = 0.$

Ans. $y^3 + 3y^2 + y + \frac{1}{3} = 0.$

2. $4x^4 + 6x^3 + 2x + 2 = 0.$

Ans. $y^4 + y^3 + 3y + 2 = 0$

3. $7x^2 - 7x - 2 = 0.$ *Ans.* $y^2 + \frac{7}{2}y - \frac{7}{2} = 0.$

The various transformations that have been explained, are of frequent use in solving equations of the form of Equation (1.)

DERIVED POLYNOMIALS AND DERIVED EQUATIONS.

201. A DERIVED POLYNOMIAL, is one that may be derived from a given polynomial, by multiplying each term by the exponent of the leading letter in that term, and then diminishing the exponent of the leading letter, by 1, in each result. The derived polynomial is generally called the *derivative;* and that from which it is derived is called the *primitive.* Thus, $x^3 + 2x^2 + 3x + 1$, being a given *primitive* polynomial, its derivative is, $3x^2 + 4x + 3$.

202. A DERIVED EQUATION, or a *derivative equation* is one whose members are derivatives of a given equation. Thus, $3x^2 + 2x + 5 = 0$, being a primitive equation its *derivative* is, $6x + 2 = 0$.

EXAMPLES.

Find the derivatives of the following equations:

1. $x^4 + 3x^2 + 2 = 0$. *Ans.* $4x^3 + 6x = 0$

2. $x^4 + 6x^3 + 20x^2 + 10 = 0$.

Ans. $4x^3 + 18x^2 + 40x = 0$.

The derivative of the general equation,

$$x^n + px^{n-1} + qx^{n-2} + \ldots + sx^2 + tx + u = 0, \quad (1.)$$

is,

$$nx^{n-1} + (n-1)px^{n-2} + (n-2)qx^{n-3} + \ldots + 2sx + t = 0. \quad (12.)$$

This equation is used in determining whether Equation (1) has equal roots.

Equal Roots.

203. We have seen that the first member of Equation (1) is composed of as many binomial factors of the form, $x - a$, $x - b$, &c., as there are roots, each of which corresponds to a root of the equation. When the equation has two roots equal to a, there will be two factors equal to $x - a$, that is, the first member will be divisible by $(x - a)^2$; when there are three roots equal to a, the first member will be divisible by $(x - a)^3$, and so on. In order to deduce a general rule for determining whether an equation has equal roots, and for freeing it of them, let us resume the general equation,

$$x^n + px^{n-1} + qx^{n-2} + \ldots + sx^2 + tx + u = 0 \ldots (1).$$

Substituting $y + r$ for x, we have,

$$(y + r)^n + p(y + r)^{n-1} + q(y + r)^{n-2} + \ldots + s(y + r)^2 + t(y + r) + u = 0.$$

Developing the different powers of $y + r$, by the Bino-

mial Formula, and arranging the result according to the ascending powers of y, we have,

$$\begin{array}{l|l|l}
y^n & y^0 + nr^{n-1} & y + \ldots + y^n = 0 \ldots (a.) \\
+ pr^{n-1} & + (n-1)pr^{n-2} & \\
+ qr^{n-2} & + (n-2)qr^{n-3} & \\
\vdots & \vdots & \\
+ sr^2 & + 2sr & \\
+ tr & + t & \\
+ u & &
\end{array}$$

In this equation, the values of y are equal to the values of x, in Equation (1), each diminished by r, since, by hypothesis, we have,

$$x = y + r\,;\ \text{whence,}\ \ y = x - r.$$

Now, r is entirely arbitrary; we may therefore assign to it a value at pleasure. If we suppose r to be a root of Equation (1), the coefficient of y^0, in Equation (a), will be equal to 0, because that coefficient is what the first member of (1) becomes, when r is substituted for x (Art. 189). Making this coefficient equal to 0, and dividing each of the remaining terms by y, we have,

$$\begin{array}{l|l}
nr^{n-1} & y^0 + \ldots + y^{n-1} = 0 \ldots (b.) \\
+ (n-1)pr^{n-2} & \\
+ (n-2)qr^{n-3} & \\
\vdots & \\
+ 2sr & \\
+ t &
\end{array}$$

Now, the $n-1$ roots of Equation (b), are equal to the different values of $x - r$, in Equation (1), that is,

to the results obtained by subtracting the root r of Equation (1), from each of the other roots in succession. If, therefore, Equation (1) has two roots equal to r, one of these differences will be equal to 0, that is, one of the roots of Equation (b), will also be equal to 0. But, when $x - r$ is equal to 0, that is, when y is equal to 0, all of the terms of Equation (b), except the first, reduce to 0, consequently that term must be separately equal to 0; or,

$$nr^{n-1} + (n-1)pr^{n-2} + (n-2)qr^{n-3} + \ldots + 2sr + t = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c.)$$

Comparing this equation with Equation (12), (Art. 202), we see that it is what the derivative of Equation (1) becomes, when for x we substitute r. Hence, we conclude that when Equation (1) has two roots equal to r, its derivative has 1 root equal to r; and conversely, when the derivative has 1 root equal to r, the primitive has 2 roots equal to r.

By a similar course of reasoning, it may be shown that, when Equation (1) has 3, 4, 5, &c., roots, each equal to r, its derivative has 2, 3, 4, &c., roots, each equal to r; and conversely, when the derivative has 2, 3, 4, &c., roots, each equal to r, the primitive has 3, 4, 5, &c., roots, each equal to r; that is, when the first member of the given equation is divisible by $(x-r)^2$, $(x-r)^3$, &c., the first member of its derivative will be divisible by $x - r$, $(x-r)^2$, &c., and the reverse.

Hence, we have the following rule to determine whether an equation has any equal roots, and to free the equation from them when any exist.

RULE.

Find the derivative equation. Then apply the rule for the greatest common divisor to the first members of the primitive and derivative equations. If no common divisor is found, the equation has no equal roots. If a common divisor is found, divide both members of the given equation by it, and the resulting equation will have no equal roots.

The operation of finding the equal roots, consists in placing the greatest common divisor found equal to 0, and solving the resulting equation. When this can be done, the equal roots may all be found. For each single root of the resulting equation, there will be two equal roots in the primitive equation; for each pair of equal roots in the resulting equation, there will be three equal roots in the given equation; and so on, as indicated in the preceding discussion. The equation found, by placing the greatest common divisor equal to 0, is to be treated, in all respects, like an original equation; and of course, the process for equal roots may be applied to it; and so on, indefinitely.

EXAMPLES.

1. Eliminate the equal roots from the equation,

$$x^5 - 27x^3 + 22x^2 + 192x - 288 = 0.$$

The derivative equation is (Art. 202),

$$5x^4 - 81x^2 + 44x + 192 = 0.$$

The greatest common divisor between the first members of the given equation and its derivative, is,

$(x-3)(x+4)$. Dividing both members of the given equation by this, there results the equation,

$$x^3 - x^2 - 14x + 24 = 0,$$

which has no equal roots.

If we place the common divisor equal to 0, we have,

$$(x-3)(x+4) = 0. \quad \therefore \; x = 3, \text{ and } x = -4.$$

Hence, the given equation has two roots equal to 3, and two roots equal to -4. Dividing both members of the given equation, by $(x-3)^2(x+4)^2$, there results,

$$x - 2 = 0. \quad \therefore \; x = 2.$$

The given equation is completely solved, and, in like manner, many other equations may be treated.

2. Eliminate the equal roots from the equation,

$$x^4 - 5x^3 + 9x^2 - 7x + 2 = 0.$$

The derivative equation is,

$$4x^3 - 15x^2 + 18x - 7 = 0.$$

The greatest common divisor of the first member is,

$$(x-1)^2.$$

Hence, the required equation is,

$$x^2 - 3x + 2 = 0;$$

which may be solved by known rules. The roots of the given equation are, 1, 1, 1, and 2.

SOLUTION OF HIGHER EQUATIONS.

204. Whenever an equation of the form (1) has equal roots, it may be freed from them by the preceding principle, and the resulting equation will be of a lower degree than the primitive one: which is always to be desired. We shall, in what follows, suppose that the equation in question has been freed of its equal roots.

It was shown, in Art. 192, that the absolute term of an equation of the form (1), is always equal to the continued product of the roots with their signs changed. When this absolute term is a whole number, and it may always be made so (Art. 195), and any root is a whole number, it may often be found by trial; for, we may resolve the absolute term into its factors, and then, by the process of synthetic division, we may see whether the first member of the given equation is exactly divisible by the unknown quantity, increased by either of these factors; if so, that factor, with its sign changed, is a root (Art. 192). If the first member is not divisible by the unknown quantity, increased by any one of the factors, then the equation has no entire roots.

EXAMPLES.

1. Find the entire root in the equation,

$$x^3 + 3x^2 + 9x - 38 = 0.$$

The divisors of -38 are, ± 1, ± 2, ± 19, ± 38.

We see, at a glance, that neither $+1$, nor -1, will satisfy the equation; hence, neither -1 nor $+1$, can

be a root. Applying the rule for synthetic division, we have,

$$\begin{array}{r|l} 1 + 3 + 9 - 38 & 2 \\ \quad + 2 + 10 + 38 & \\ \hline 1 + 5 + 19, + 0. & \end{array}$$

Hence, the first member is divisible by $x - 2$, and $+ 2$ is a root. Dividing both members of the given equation by $x - 2$, we have,

$$x^2 + 5x + 19 = 0;$$

which can be solved by known rules. Both of its roots are imaginary, one equal to $-\frac{5 + \sqrt{-51}}{2}$, and $\frac{-5 - \sqrt{-51}}{2}$.

2. Find the entire root of the equation,

$$x^3 - 12x^2 + 4x + 207 = 0.$$

The divisors of 207 are,

$$\pm 1, \pm 3, \pm 9, \pm 23, \pm 69, \text{ and } \pm 207.$$

Testing each divisor, we find that the first member of the given equation is divisible by $x - 9$. Hence, 9 is the root required.

When freed of this root, the given equation becomes,

$$x^2 - 3x - 23 = 0,$$

which gives the roots, $\frac{3 + \sqrt{101}}{2}$, and $\frac{3 - \sqrt{101}}{2}$.

In the same way, the following equations may be solved:

3. $x^3 + 3x^2 - 6x - 8 = 0.$

Ans. $x' = 2$, $x'' = -1$, and $x''' = -4$.

4. $x^3 + 9x - 1430 = 0.$

$$x' = 11,\ x'' = \frac{-11+\sqrt{-399}}{2},\ \text{and}\ x'' = \frac{-11-\sqrt{-399}}{2}.$$

5. $x^3 - 1 = 0.$

$$x' = 1,\ x'' = \frac{-1+\sqrt{-3}}{2},\ \text{and}\ x'' = \frac{-1-\sqrt{-3}}{2}.$$

An equation of the form (1), whose coefficients are entire, has no roots that are irreducible fractions. For, if so, suppose one of these roots to be of the form $\frac{a}{b}$, an irreducible fraction.

Substituting this fraction for x, in Equation (1), and transposing all of the terms, except the first, to the second member, we have,

$$\frac{a^n}{b^n} = -p\frac{a^{n-1}}{b^{n-1}} - q\frac{a^{n-2}}{b^{n-2}} - \ldots - r\frac{a}{b} - u.$$

And multiplying both members by b^{n-1}, we have,

$$a^{n-1} \times \frac{a}{b} = -pa^{n-1} - qa^{n-2}b - \ldots - rab^{n-2} - ub^{n-1}.$$

The first member of this equation is an irreducible fraction, and the second member is entire: which is manifestly impossible. Hence, the supposition that a root can be an irreducible fraction, is absurd.

IMAGINARY ROOTS.

205. The imaginary roots of an equation are of the form, $a \pm b\sqrt{-1}$.

It may be shown that imaginary roots always enter by pairs, that is, if $a + b\sqrt{-1}$, is a root, then will $a - b\sqrt{-1}$, be a root, also. For, substituting $a + b\sqrt{-1}$ for x, in Equation (1), we have,

$$(a + b\sqrt{-1})^n + p(a + b\sqrt{-1})^{n-1} + \ldots + t(a + b\sqrt{-1}) + u = 0; \quad \ldots \quad (a.)$$

If we develop the different powers of $a + b\sqrt{-1}$, by the Binomial Formula, and perform the operations indicated, the first member will be made up of two kinds of terms, *real* and *imaginary*. The imaginary terms arise from the odd powers of $b\sqrt{-1}$, and consequently (Art. 138), they contain no other imaginary factor than $\sqrt{-1}$. Hence, denoting the sum of the real quantities by m, and the sum of the coefficients of the factor, $\sqrt{-1}$, by n, the preceding equation becomes,

$$m + n\sqrt{-1} = 0 \quad . \quad . \quad . \quad . \quad (b.)$$

But, from Principle 2, Art. 138, Equation (b) can only be true, by making $m = 0$, and $n = 0$.

If we substitute $a - b\sqrt{-1}$, for x, in Equation (1), and perform the operations indicated, the result will differ from that expressed by Equation (b), in the signs of the odd powers of $b\sqrt{-1}$. Hence, the resulting equation will be,

$$m - n\sqrt{-1} = 0 \quad . \quad . \quad . \quad . \quad (c.)$$

But, we have shown that $m = 0$, and $n = 0$; hence, Equation (c) is satisfied; and consequently, the substitution of $a - b\sqrt{-1}$ for x, in Equation (1), satisfies it, which shows that $a - b\sqrt{-1}$ is a root.

Now, if $a + b\sqrt{-1}$, is a root of Equation (1), the first member must be divisible by $x - (a + b\sqrt{-1})$; and from what has just been proved, it must also be divisible by $x - (a - b\sqrt{-1})$; and consequently, by the product of these factors. But, the product of these two factors is always positive for all real values of x (Principle 3, Art. 138). We may, therefore, conclude that *the number of imaginary roots is always even*, since they enter by pairs; and that the product of all

the binomial factors, corresponding to imaginary roots, will remain positive for all real values of x.

An easy method of determining the number of real roots of an equation, and consequently, the number of imaginary roots, is due to STURM, and is called STURM'S THEOREM. The limits of the real roots may also be determined by this theorem.

STURM'S THEOREM.

206. If we denote the real roots of Equation (1), by a, b, c, &c., and suppose them arranged according to their values, so that a shall be the smallest algebraically, that is, nearest to $-\infty$, b the next smallest, and so on; and if we denote the product of all the factors corresponding to imaginary roots, which is always positive, by Y, Equation (1) may be written,

$$(x-a)(x-b)(x-c)\ldots Y=0 \quad . \quad . \quad (13.)$$

If we suppose $x=-\infty$, each of the factors $(x-a)$, $(x-b)$, &c., will be negative; and the first member will be negative, when there are an odd number of real roots, and positive when there are an even number. If we suppose x to increase from $-\infty$ towards $+\infty$, assuming in succession every possible value, the sign of the first member of (13), will remain unchanged, until x becomes equal to a, when the first member becomes equal to 0. For all values of x between, a and b, the factor $x-a$ will be positive, and all the remaining ones will be negative; hence, the product of all the factors, or the first member of (13), will have a different sign from what it had before, that is, the first member changes its sign from $+$ to $-$, or from $-$ to $+$, when the value of x passes over the real root a. When x

becomes equal to b, the first member again becomes 0. For all values of x between b and c, the first two factors are positive, and all the remaining ones negative. Hence, in this case, the sign of the first member is the same as in the first instance, that is, the sign of the first member changes from $-$ to $+$, or from $+$ to $-$, when the value of x passes over the real root b. In the same way it may be shown, that the sign of the first member changes from $+$ to $-$, or from $-$ to $+$, as often as the value of x passes over a real root, and that it does not change in any other case. If in any equation of the form (1), we suppose x to assume every possible value, from $-\infty$ up to $+\infty$, and determine the number of times that the first member changes sign, we shall have the number of real roots, and consequently the number of imaginary roots in the equation. The object of STURM'S THEOREM is, to show the manner of determining the number of such changes of sign.

The demonstration of STURM'S THEOREM depends upon a few simple principles.

207. Let the first member of any equation of the form (1), after having been freed from its equal roots, (Art. 204), be denoted by X, and let the first derived polynomial of X, be denoted by X_1. If we apply to X and X_1, the process for finding their greatest common divisor, differing only in this respect, that instead of using the successive remainders, as obtained, *we change their signs*, and also take care neither to introduce nor reject any factor except a *positive one*, we shall ultimately arrive at a remainder independent of x. There will be no common divisor, for the reason that we have supposed

the equation, $X = 0$, to have been freed of its equal roots (Art. 204). Let us denote the several remainders, with their signs changed, by X_2, X_3, ... X_r, and the corresponding quotients by Q_1, Q_2, Q_3 ... Q_r. From the nature of the process, we shall have the following equations:

$$\begin{aligned} X &= X_1 Q_1 - X_2 \\ X_1 &= X_2 Q_2 - X_3 \\ &\cdots\cdots \\ X_{n-1} &= X_n Q_n - X_{n+1} \\ &\cdots\cdots \\ X_{r-2} &= X_{r-1} Q_{r-1} - X_r. \end{aligned}$$

Now, it is evident that no two consecutive values of the expressions, X, X_1, X_2, &c., can become 0, for the same value of x; for, suppose that for $x = a$, we have, X_{n-1}, and X_{n-2}, both equal to 0, then, from the relation between these expressions and X_{n+1}, the latter expression will also be equal to 0; and from the next equation, the expression, X_{n+2}, will then be equal to 0, and so on to the last equation, which will give X_r, also equal to 0, which is impossible. Of course, X and X_1 cannot both be 0, for the same value of x, since the equation $X = 0$ has been freed of its equal roots. Hence, the following principle:

1. *If any numbers be substituted for x, in the expressions, X, X_1, X_2, &c., it is impossible that any two consecutive ones can reduce to* 0 *at the same time.*

By examining the preceding group of equations, we see that if any value of x reduces one of the expressions to 0, the preceding and succeeding ones must have contrary signs; thus, if $x = a$ makes X_n equal to 0, then, for the same value of x, the values of X_{n-1},

and X_{n+1}, must have contrary signs. Hence, the following principle:

2. *If any number be substituted for x, in the expressions, X, X_1, &c., which reduces one of them to 0, the preceding and following ones will have contrary signs for the same value of x.*

If we denote the value of X, when we make x equal to $r + y$, by the symbol $(X)_{x=r+y}$, we shall have, from Equation (a), Art. 204,

$$(X)_{x=r+y} = A + A'y + A''y^2 + \&c.,$$

in which A denotes what X becomes, when x is replaced by r; A', what X' becomes, when x is replaced by r, &c. Now, if we suppose r to be a root of the equation, $X = 0$, we shall have $A = 0$, and and the preceding equation becomes,

$$(X)_{x=r+y} = y(A' + A''y + \&c.).$$

Now, it is plain, that a value may be given to y so small that the sum of all the terms within the parenthesis after A', shall be less than A'; for, if y be made inappreciably small, then will $A''y$ be inappreciably small in comparison with A', which is finite; and since the following terms contain higher powers than the first, each will be inappreciably small with respect to $A''y$; and because the number of terms is finite, the sum of all after the first must be less than the first: which was to be shown. Hence, when y is inappreciably small, the sum of the terms within the parenthesis must have the same sign as A'. Now, if we make y inappreciably small, first positive, and then negative, we shall have,

$$(X)_{x=r+y} = +y(A' + A''y + \&c.)$$

and

$$(X)_{x=r-y} = -y(A' + A'' + y\,\&c.)$$

When y is positive, the sign of the second member will be the same as that of A', and consequently the sign of the first member must be the same as that of A'; but when y is negative, just the reverse will obtain. Hence, the following principle:

3. *If a value for x, inappreciably smaller than one of the real roots of the equation, $X = 0$, be substituted for x, in the expressions, X, X_1, X_2, &c., the corresponding values of X, and its derivative X_1, will have the contrary signs; and if we substitute for x, a value inappreciably greater than one of the real roots, the signs of X and X_1 will be the same.*

208. Let us arrange the expressions, X, X_1, X_2, ... X_n ... X_r, in a row, and then substitute in each the same value for x, writing beneath each the resulting sign. Some of these signs may be positive and some negative, and the positive and negative signs may occur in any order. Now, in passing along the row, whenever two consecutive signs are alike, there is said to be a *permanence*, and whenever two consecutive signs are unlike, there is said to be a *variation*.

Let us next see what changes take place in the number of permanences and variations, when x is made to increase, by inappreciably small increments, from $-\infty$ up to $+\infty$; we shall find, in succession, values of x, which reduce some of the expressions to 0; but, from Principle 1, no value will be met with that will reduce two consecutive ones to 0 at the same time.

Now, there may be two cases: *first*, when X reduces to 0; and *secondly*, when any one of the other expressions reduces to 0. When X reduces to 0, the corresponding value of x is a real root of the equation, $X = 0$; for

the value of x immediately preceding this root, X and X_1 have contrary signs, and there is a *variation;* for the value of x immediately following this root, X and X_1 have the same sign, giving a permanence. Hence, when x passes over a real root, there is a variation lost, or changed into a permanence. When any of the expressions, X_1, X_2, &c., reduce to 0, the preceding and following ones have contrary signs (Principle 2) and there is a variation. Now, for the values of x immediately preceding and following this one, the expressions immediately preceding and following the one that reduces to 0, will still have contrary signs; and whatever may be the sign of the middle one, there will still be one variation in passing over the three, and but one. Hence, we conclude that there is no variation either lost or gained, when x passes over a value that reduces any of the expressions after X to 0. Taking the whole row of signs together, there will be one variation lost, and only one, every time that the value of x passes one of the real roots of the equation. Hence,

If we substitute, in the first place, for x, $-\infty$, *and then* $+\infty$, *writing down the corresponding signs in a row, the number of variations in the first row, diminished by the number of variations in the second row, will be equal to the number of real roots in the given equation.*

This is STURM'S theorem.

As an example, let it be required to find the number of real roots in the equation,

$$x^4 - 12x^2 + 12x - 3 = 0.$$

Here, we have, $X = x^4 - 12x^2 + 12x - 3$. Finding

the derivative of X, and suppressing the factor, $+4$, we have, $X_1 = x^3 - 6x + 3$. Dividing X by X_1, we find, for a remainder, $-6x^2 + 9x - 3$; suppressing the factor, $+3$, and changing the sign of the result, we have, $X_2 = 2x^2 - 3x + 1$. Multiplying X_1 by 4, and dividing by X_2, we have, for a remainder, $-17x + 9$; changing the sign, we have, $X_3 = 17x - 9$. Multiplying X_2 by 289, and dividing by X_3, we find, for a remainder, -8; changing the sign, we have, $X_4 = 8$. Writing these expressions in a row, and substituting $-\infty$, and then $+\infty$ for x, we have the results indicated below.

	$x = -\infty$.	$x = +\infty$.
$X = x^4 - 12x^2 + 12x - 3$	+	+
$X_1 = x^3 - 6x + 3$	−	+
$X_2 = 2x^2 - 3x + 1$	+	+
$X_3 = 17x - 9$	−	+
$X_4 = 8$	+	+

In the first case, there are 4 variations, and in the second, there are no variations; hence, all four of the roots are real.

Sturm's Theorem also enables us to determine the places of the real roots. If we substitute for x, in the above expressions, any two numbers whatever, the number of variations corresponding to the lesser, diminished by that corresponding to the greater, will give the number of real roots between the two numbers. Let us begin by making x equal to 0, 1, 2, &c., until we get as many permanences, as when $x = \infty$; then make x equal to -1, -2, &c., until we get as many variations, as when $x = -\infty$. We shall have the following results:

	$x=0$,	$x=1$,	$x=2$,	$x=3$,	$x=4$,	$x=-1$,	$x=-2$,	$x=-3$
X ,	−	−	−	−	+	−	−	−
X_1 ,	+	−	−	+	+	+	+	−
X_2 ,	+	0	+	+	+	+	+	+
X_3 ,	−	+	+	+	+	−	−	−
X_4 ,	+	+	+	+	+	+	+	+

Hence, we conclude that two of the roots lie between 0 and + 1; one between 3 and 4; and one between − 3 and − 4. Here, we have found the values of the roots, to within less than 1. The method of completing the approximation, will be explained hereafter.

In the preceding case, we have seen that + 4 gives the same number of permanences, as + ∞; hence, no real roots lie between + 4 and + ∞. We have also seen, that − 4 and − ∞, give the same number of variations; hence, no real root lies between them. The values, − 4 and + 4, are called the LIMITS of the roots of the given equation; the former being the inferior, and the latter, the superior limit.

If we consider the positive roots alone, 0 and 4 are the limits. If we consider the negative roots alone, − 4 and − 3 are the limits. In the same manner, the limits of the positive and negative roots of any equation may be found. It is often useful to determine these limits, especially when seeking the entire roots of an equation, by the process of Article 205.

EXAMPLES.

1. Find the number, the places, and the limits of the real roots of the equation, $x^3 + x^2 + x - 100 = 0$.

OPERATION.

$$X = x^3 + x^2 + x - 100$$
$$X_1 = 3x^2 + 2x + 1$$
$$X_2 = -4x + 901$$
$$X_3 = -2442627$$

	$x=-\infty$		$x=+\infty$	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$
X	$-$		$+$	$-$	$-$	$-$	$-$	$-$	$+$
X_1	$+$		$+$	$+$	$+$	$+$	$+$	$+$	$+$
X_2	$+$		$-$	$+$	$+$	$+$	$+$	$+$	$+$
X_3	$-$		$-$	$-$	$-$	$-$	$-$	$-$	$-$

Hence, there is one real root lying between 4 and 5, which are the limits of the root.

2. Find the number, places, and limits of the real roots of the equation,

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

Ans. There are 4 real roots; one between 0 and 1, one between 2 and 3, one between 5 and 6, and one between -1 and 0. The limits are -1 and 6.

3. Find the number, places, and limits of the real roots of the equation, $x^3 - 23x - 24 = 0$.

Ans. 3 real roots, one between 5 and 6, one between -1 and -2, and one between -4 and -5. The limits are, -5 and $+6$.

4. Find the number, places, and limits of the real roots of the equation, $x^3 + \frac{3}{2}x^2 - 2x - 5 = 0$.

Ans. There is 1 real root, and it lies between the limits 1 and 2.

APPENDIX.

1. It is proposed to demonstrate, in this Appendix, several useful principles; such as those employed in factoring, the Binomial Theorem for any exponent, the method of summing Series, and the like. These principles are all of high importance in the more advanced portions of the mathematical course; but, on account of their difficulty, it has not been thought best to introduce their demonstration into the body of the work.

Principles employed in Factoring.

I. *The difference of the like powers of any two quantities, is always divisible by the difference of the quantities.*

2. To demonstrate this principle, let a and b denote any two quantities whatever, and m any positive whole number whatever; then will $a^m - b^m$ denote the difference between the like powers of any two quantities, and $a - b$ the difference between the quantities: If we commence the division by the rule, we shall have the following operation:

$$\begin{array}{l|l} a^m - b^m & a - b \\ a^m - a^{m-1}b & \overline{a^{m-1}} \\ \hline a^{m-1}b - b^m & \end{array}$$

The remainder may be factored, and placed under the form,

$$b(a^{m-1} - b^{m-1}).$$

Hence, we have the result,

$$\frac{a^m - b^m}{a - b} = a^{m-1} + b\,\frac{a^{m-1} - b^{m-1}}{a - b} \quad \cdots (1.)$$

The second member of Equation (1) will be entire, and consequently, the first member will be entire, when $\frac{a^{m-1} - b^{m-1}}{a - b}$ is entire: that is, *if the difference of the* $(m-1)^{th}$ *powers of two quantities is divisible by the difference of the quantities, then will the difference of the* m^{th} *powers of the two quantities also be divisible by the difference of the quantities.*

But, we know that the difference of the second powers is divisible by the difference of the quantities, giving for a quotient the sum of the quantities; hence, from the principle above demonstrated, the difference of the cubes is also divisible by the difference of the quantities. It having been proved that the difference of the cubes is divisible, it follows, from the principle demonstrated, that the difference of the fourth powers is also divisible by the difference of the quantities. The difference of the fourth powers being divisible, it follows, as before, that the difference of the fifth powers is divisible; and so on, by successive deduction, it may be shown that the division is possible when m is any positive number whatever. Hence, the principle is proved.

We found the first term of the quotient to be, a^{m-1}; and if we perform a second partial division, we shall get, for the second term of the quotient, $a^{m-2}b$, with a second remainder, $b^2(a^{m-2} - b^{m-2})$; dividing again, we shall

get, for a third term of the quotient, $a^{m-3}b^2$; and so on. We see that the exponent of a goes on diminishing by 1, in each term of the quotient, to the right; whilst the exponent of b, being 0, in the first term, goes on increasing by 1, in each term, to the right: the last term of the quotient contains a to the 0 power, and b to the $m-1$ power. Hence, we may write the quotient as follows:

$$\frac{a^m - b^m}{a - b} = a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \ldots + a^2b^{m-3} + ab^{m-2} + b^{m-1} \quad \ldots . \quad (2.)$$

If, in Equation (2), we replace a by c^2, and b by d^2, we shall have,

$$\frac{c^{2m} - d^{2m}}{c^2 - d^2} = c^{2m-2} + c^{2m-4}d^2 + \ldots + c^2d^{2m-4} + d^{2m-2} \quad (3.)$$

Whatever may be the value of m, $2m$ will be an even number, and the second member of (3) will be entire; hence, we conclude that,

II. *The difference of like even powers of any two quantities is always divisible by the difference of the squares of the quantities.*

3. If we multiply both members of (3), by $(c - d)$, we shall have

$$\frac{c^{2m} - d^{2m}}{c + d} = (c-d)\,(c^{2m-2} + c^{2m-4}d^2 + \ldots + c^2d^{2m-4} + d^{2m-2} \quad (4.)$$

The second member of (4) is entire, consequently the first member is also entire. Hence,

III. *The difference of like even powers of any two quantities, is always divisible by the sum of the quantities.*

4. If, in Equation (1), we replace a by c^n, and b by d^n, we shall have,

$$\frac{c^{mn} - d^{mn}}{c^n - d^n} = c^{mn-n} + c^{mn-2n}d^n + \ldots + d^{mn-n} \quad (5.)$$

Whatever may be the value of m, the second member of (5) will be entire. Hence,

IV. *The difference of like powers of two quantities, is always divisible by the difference of any other like powers of the two quantities, whenever the exponent in the first case is exactly divisible by that in the second case.*

5. Let us next ascertain, under what circumstances the sum of the like powers of two quantities is divisible by the sum of the quantities.

Commencing the operation, we have,

$$\begin{array}{l|l} a^m + b^m & a+b \\ \underline{a^m + a^{m-1}b} & a^{m-1} \\ -a^{m-1}b + b^m & \end{array}$$

Factoring the remainder, and writing it over the divisor, as a fraction, we have,

$$\frac{a^m + b^m}{a+b} = a^{m-1} - b\left(\frac{a^{m-1} - b^{m-1}}{a+b}\right). \quad (6.)$$

Now, the second member of (6), and consequently the first member, will be entire, when $a^{m-1} - b^{m-1}$, is divisible by $a+b$; but from Principle 3, this will be the case, when $m-1$ is an even number; that is, when m is an odd number. Hence, the following principle:

V. *The sum of like odd powers of any two quantities is always divisible by the sum of the quantities.*

6. It will be seen, by inspection, that the form of the quotient is the same as that of $a^m - b^m$ by $a - b$, except that the terms are alternately plus and minus; that is,

$$\frac{a^m + b^m}{a + b} = a^{m-1} - a^{m-2}b + a^{m-3}b^2 \ldots - ab^{m-2} + b^{m-1} \qquad (7.)$$

An example of the application of each principle is given below.

1. $\dfrac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$

2. $\dfrac{a^6 - b^6}{a^2 - b^2} = a^4 + a^2b^2 + b^4.$

3. $\dfrac{a^6 - b^6}{a + b} = (a - b)\,(a^4 + a^2b^2 + b^4).$

4. $\dfrac{a^9 - b^9}{a^3 - b^3} = a^6 + a^3b^3 + b^6.$

5. $\dfrac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$

7. It has been shown that, when m is a positive whole number, we have,

$$\frac{a^m - b^m}{a - b} = a^{m-1} + ba^{m-2} + \ldots + ab^{m-2} + b^{m-1},$$

in which the quotient has m terms. This is true for all values of a and b; hence, it will be true when $a = b$. In this case, $\dfrac{a^m - b^m}{a - b}$, is a vanishing fraction (Art. 75). Denoting what the first member becomes, when $a = b$, by the symbol, $\left(\dfrac{a^m - b^m}{a - b}\right)_{a=b}$, and recollecting that each of the terms of the quotient reduces to a^{m-1}, we shall have, since there are m terms in the quotient,

$$\left(\frac{a^m - b^m}{a - b}\right)_{a=b} = ma^{m-1} \quad . \quad . \quad . \quad (1.)$$

We propose to show, that Equation (1) is true, whatever may be the value of m; that is, whether m be positive or negative, entire or fractional.

First. Suppose m to be a positive fraction, equal to $\frac{p}{q}$.

Make $a^{\frac{1}{q}} = x$, and $b^{\frac{1}{q}} = y$; whence, $a^{\frac{p}{q}} = x^p$, $b^{\frac{p}{q}} = y^p$, $a = x^q$, and $b = y^q$.

We shall have,

$$\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b} = \frac{x^p - y^p}{x^q - y^q}.$$

Dividing both terms of the second member, by $x - y$, we have,

$$\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b} = \frac{\left(\frac{x^p - y^p}{x - y}\right)}{\left(\frac{x^q - y^q}{x - y}\right)} \quad . \quad . \quad . \quad (2.)$$

If we make $a = b$, we shall have, $x = y$; in which case, the numerator of the second member of (2) becomes, px^{p-1}; and the denominator becomes qx^{q-1}, by the preceding principle.

$$\left(\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b}\right)_{a=b} = \frac{px^{p-1}}{qx^{q-1}} = \frac{p}{q}x^{p-q};$$

Or, substituting for x, its value, $a^{\frac{1}{q}}$, and reducing,

$$\left(\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b}\right)_{a=b} = \frac{p}{q}a^{\frac{p}{q}-1} \quad . \quad . \quad . \quad . \quad (3.)$$

Hence, *the form is true for all positive values of the exponent.*

Secondly. Suppose m to be negative, and either entire or fractional; that is, let $m = -n$. We shall have,

$$\frac{a^{-n} - b^{-n}}{a - b} = -a^{-n}b^{-n} \times \frac{a^n - b^n}{a - b}, \quad (4.)$$

Since, $a^{-n} - b^{-n} = -a^{-n}b^{-n} \times (a^n - b^n.)$

Now, if we suppose $a = b$, the first factor of the second member of (4) reduces to $-a^{-2n}$; and the second factor reduces to na^{n-1}, by the preceding principle. Hence,

$$\left(\frac{a^{-n} - b^{-n}}{a - b}\right)_{a=b} = -a^{-2n} \times na^{n-1} = -na^{-n-1}. \quad (5.)$$

Hence, *the form is true when the exponent is negative, and either entire or fractional.* It is, therefore, general; that is, the equation,

$$\left(\frac{a^m - b^m}{a - b}\right)_{a=b} = ma^{m-1},$$

is true for all values of m, positive or negative, entire or fractional.

This principle is employed in demonstrating the Binomial Theorem, in the general case.

General Demonstration of the Binomial Theorem.

8. The object of the Binomial Theorem is to show how to develop any power of a binomial into a series.

Let us assume the development,

$$(1+z)^m = P + Qz + Rz^2 + Sz^3 + \&c. \quad . \quad (1.)$$

It is required to find such values for P, Q, R, &c., as

shall make Equation (1) true for all values of m and z, that is, such values as will make Equation (1) identical (Art. 174).

Since Equation (1) is to be true for all values of z, it must be so when $z = 0$. Making $z = 0$, in (1), we have,

$$1 = P, \text{ or, } P = 1.$$

Substituting this value of P in (1), it becomes,

$$(1 + z)^m = 1 + Qz + Rz^2 + Sz^3 + \&c. \quad . \quad . \quad (2.)$$

Since (2) is to be true for all values of z, we may replace z by y, giving,

$$(1 + y)^m = 1 + Qy + Ry^2 + Sy^3 + \&c. \quad . \quad . \quad (3.)$$

Subtracting (3) from (2), member from member, we have,

$$(1+z)^m - (1+y)^m = Q(z-y) + R(z^2-y^2) + \&c. \quad . \quad (4.)$$

Dividing the first member of (4), by $(1 + z) - (1 + y)$, and the second by its equal, $z - y$, we have,

$$\frac{(1+z)^m - (1+y)^m}{(1+z) - (1+y)} = Q + R\frac{z^2-y^2}{z-y} + S\frac{z^3-y^3}{z-y} + \&c. \quad (5.)$$

If, in (5), we make $1 + z = 1 + y$, whence $z = y$, in accordance with the principle demonstrated in the preceding article, we shall have,

$$\left(\frac{(1+z)^m - (1+y)^m}{(1+z) - (1+y)}\right)_{1+z=1+y} = m(1+z)^{m-1},$$

$$\left(\frac{z^2-y^2}{z-y}\right)_{z=y} = 2z, \quad \left(\frac{z^3-y^3}{z+y}\right)_{z=y} = 3z^2, \text{ \&c.}$$

Hence, Equation (5), under this hypothesis, becomes,

$$m(1 + z)^{m-1} = Q + 2Rz + 3Sz^2 + \&c. \quad . \quad . \quad (6.)$$

Multiplying both members of (6), by $1 + z$, it becomes,

$$m(1+z)^m = \begin{array}{l|l|l} Q + 2R & z + 3S & z^2 + \text{\&c.} \\ + \ Q & + 2R & \end{array} \quad . \ . \quad (7.)$$

Multiplying both members of (2), by m, it becomes,

$$m(1+z)^m = m + mQz + mRz^2 + mSz^3 + \text{\&c.} \quad . \ . \quad (8.)$$

The first members of (7) and (8) are equal, hence, their second members are also equal, hence,

$$m + mQz + mRz^2 + \text{\&c.} = \begin{array}{l|l|l} Q + 2R & z + 3S & z^2 + \text{\&c.} \\ + \ Q & + 2R & \end{array} \quad (9.)$$

Equation (9) is identical, that is, it is true for all values of m and z; hence, from the principle of indeterminate coefficients (Art. 176), the coefficients of the like powers of z, in the two members, are equal. Equating these coefficients, we have,

$$\begin{aligned} m &= Q & \therefore \ Q &= m \\ mQ &= 2R + Q & \therefore \ R &= \frac{m(m-1)}{1.2} \\ mR &= 3S + 2R & \therefore \ S &= \frac{m(m-1)\,(m-2)}{1.2.3}. \end{aligned}$$

The law is apparent.

Substituting these values in (2),

$$(1+z)^m = 1 + mz + \frac{m.m-1}{1.2}z^2 + \frac{m.(m-1)\,(m-2)}{1.2.3}z^3 + \text{\&c.} \quad (10.)$$

If, in (10), we replace z by $\frac{b}{a}$, and then multiply both members by a^m, and reduce, we shall have,

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m \cdot m-1}{1 \cdot 2}a^{m-2}b^2 + \frac{m(m-1)\,(m-2)}{1 \cdot 2 \cdot 3}a^{m-3}b^3 + \&c., \quad (11.)$$

which is the Binomial Formula, and is true for any value of m, either positive or negative, entire or fractional.

By comparing Equation (11) with that of Art. 104, we see that they are of the same form; hence, the formula deduced in Art. 104 is general.

A few examples are annexed, to illustrate the application of the formula above deduced.

Let it be required to develop the following expressions:

1. $\sqrt{a-x}$; or, $(a-x)^{\frac{1}{2}}$.

   Ans. $\sqrt{a}\left(1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{3x^3}{48a^3} - \frac{15x^4}{384a^4} - \&c.\right)$.

2. $(a^2+x^2)^{\frac{3}{2}}$.

   Ans. $a^3 + \frac{3ax^2}{2} + \frac{3x^4}{8a} - \frac{3x^6}{48a^3} + \frac{9x^8}{384a^5} - \&c.$

3. $\frac{a^3}{(a-x)^3}$; or, $a^3 \times (a-x)^{-3}$.

   Ans. $1 + \frac{3x}{a} + \frac{6x^2}{a^2} + \frac{10x^3}{a^3} + \&c.$

4. $\frac{1}{\sqrt{1+x}}$; or, $(1+x)^{-\frac{1}{2}}$.

   Ans. $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \&c.$

5. $(3 - 7x)^{-\frac{1}{3}}$.

Ans. $\frac{1}{\sqrt[3]{3}}\left(1 + \frac{7x}{9} + \frac{2}{9}\left(\frac{7}{3}\right)^2 x^2 + \frac{14}{81}\left(\frac{7}{3}\right)^3 x^3 + \text{\&c.}\right)$.

Summation of Series.

9. The SUMMATION of a series is the operation of finding an expression for the sum of any number of terms of the series. The method of greatest general utility, is that by *differences*. Let a, b, c, d, &c., represent the successive terms of any series. If we subtract each term of this series from the following one, we shall thus form a new series, which is called the *first order of differences*. If, in like manner, we subtract each term of this series from the following one, we shall form a new series, called the *second order of differences*; and so on, as exhibited below.

$a,\quad b,\quad c,\quad d,\quad e,$ &c.

$b-a,\quad c-b,\quad d-c,\quad e-d$ *1st order of diff.*

$c-2b+a,\quad d-2c+b,\quad e-2d+c$ *2d order of diff.*

If we designate the first terms of the first, second, &c., orders of differences by d_1, d_2, &c., we shall have,

$d_1 = b - a \qquad \therefore\ b = a + d_1$

$d_2 = c - 2b + a \qquad \therefore\ c = a + 2d_1 + d_2$

$d_3 = a - 3c + 3b - a \qquad \therefore\ d = a + 3d_1 + 3d_2 + d_3$

If we designate that term of the given series which has n terms before it, by T, we shall find, by continuing the operation,

$$T = a + nd_1 + \frac{n.(n-1)}{1.2}d_2 + \frac{n(n-1)\ (n-2)}{1.2.3}d_3, +, \qquad (1.)$$

This formula enables us to find the $(n+1)^{th}$ term, when we know the first term, and a sufficient number of the first terms of the successive orders of differences.

Let us take the series,

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c.,\ .\ .\ .\quad (2.)$$

The first order of differences of the series (2) is the series,

$$a,\ b,\ c,\ d,\ \&c.,\quad .\ .\ .\ .\ .\quad (3.)$$

Now, it is clear that the sum of n terms of the series (3) is equal to the $(n+1)^{th}$ term of the series (2); but the first term of the first order of differences is a, the first term of the second order of differences is d, and so on. If, therefore, we denote the sum of n terms of series (3) by S, which is the same thing as the $(n+1)^{th}$ term of (2), we shall have the value of S from (1), by making $a=0$, $d_1=a$, $d_2=d_1$, $d_3=d_2$, &c. Hence,

$$S = na + \frac{n(n-1)}{1.2}d_1 + \frac{n(n-1)(n-2)}{1.2.3}d_2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3 + \&c. \quad (4).$$

When all the terms of any order of differences become equal, each term of every succeeding order of differences becomes 0, and the formula gives the exact sum of the terms. When the terms of no order of differences become equal, the formula only gives approximate results, which will be the nearer true, the greater the number of terms employed.

EXAMPLES.

1. Find the sum of n terms of the series,

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\ \&c.$$

OPERATION.

Series,	1,	4,	9,	16,	25,	36,	&c.
1*st order of diff.*	3,	5,	7,	9,	11,	&c.	
2*d order of diff.*	2,	2,	2,	2,	&c.		
3*d order of diff.*	0,	0,	0,	&c.			

Here, $a = 1,\ d_1 = 3,\ d_2 = 2,\ d_3 = 0,\ d_4 = 0,$ &c.

Hence, by substitution in (4),

$$S = n + \frac{n(n-1)}{1.2} \times 3 + \frac{n(n-1)\,(n-2)}{1.2.3} \times 2.$$

2. Find the sum of 8 terms of the series,

$$1,\quad 1+2,\quad 1+2+3,\quad 1+2+3+4,\quad \&c.$$

Here, the first order of differences is,

$$2,\quad 3,\quad 4,\quad \&c.$$

And the second order of differences is,

$$1,\quad 1,\quad 1,\quad \&c.$$

The succeeding orders being each made up of zeros.

Here, $n = 8,\ a = 1,\ d_1 = 2,\ d_2 = 1,\ d_3 = 0,$ &c.; hence,

$$S = 8 + \frac{8\times 7}{1.2} \times 2 + \frac{8\times 7\times 6}{1.2\times 3} \times 1 = 8+56+56 = 120.$$

3. Find the sum of 10 terms of the series,

$$1^3,\ 2^3,\ 3^3,\ 4^3,\ 5^3,\ \&c. \qquad \textit{Ans.}\ S = 3025.$$